From Mathematics to modelling

June 6, 2015

1 Introduction

Suppose that a great mathematician, e.g. Burkard Alphers, is giving a lecture on how modelling can be used as a didactic method for teaching mathematics for engineers (or other mathematics users). Suppose then that you are asked to give a response lecture (?).

As a pure mathematician teaching engineers, I answer this problem by stating some selected experienced questions from my students. Then I answer the students questions for you by giving a lecture from a pure mathematician.

Based on the lecture which includes the answer of the questions, I give some additional question for debate.

Question 1. *Does my background in mathematics make me an enthusiastic lecturer?*

Question 2. *Is the deeper knowledge necessary for a lecturer, or just nice?*

Question 3. *Is the mathematical knowledge a disadvantage for teaching the engineering students? That is, are the toolmakers unable to teach using the tools?*

Question 4. *Why are there no questions about didactics? (Just give it to me!)*

2 Questions from good math students

Question 5. *I like these complex numbers, but do we really need them?*

For example, when we do mathematical modelling, when we end up with complex solutions to the characteristic equation, even then we are only interested in the real solutions. By the way:

Question 6. *When we solve 2. order differential equations (homogeneous with constant real coefficients). Then you guess on two (linearly independent) solutions (you'r so smart), but how do you know that there are no more?*

3 The usual answer to the question

The answer to question 5 is usually that complex numbers are needed in computations, just as negative numbers are. For example, when you and your fellow share a bill on 40 dollars, you should pay 40+(-20) kroner. The number -20 doesn't exist in real life, it is just needed for computations.

My personal favorite is the following: You are walking on the street with 20 kr. in your pocket. Then a robber comes along and steals 30 kr. from you. Then you owe the robber 10 kr.

Then the answer to question 6 should usually be that the dimension of the space of solutions to (n'th order homogeneous with constant real coefficients) differential equations are of dimension n, which means that if you can guess n linearly independent functions that each of them is a solution to the equation, then every solution is a linear sum of these.

Question 7. *Can you be a bit more specific, that is, this sounds interesting. Can you tell the story?*

4 The Story about Solving Differential Equations

Of course, you have now seen where differential equations come into the picture when we are modelling real life situations. But the story starts long before that.

4.1 Introducing the characters in the story

Definition 1. *The complex numbers are the set* $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$ *with the obvious addition and multiplication, using the rule that* $i^2 = -1 = -1 + 0i$.

Notice that a real number a is also complex, because $a = a + 0i$. So the complex numbers contain the reals: $\mathbb{R} \subset \mathbb{C}$.

Definition 2. *A polynomial with complex coefficients is an expression on the form*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

with all $a_i \in \mathbb{C}$. *The degree of* $f(x)$ *is* n, *supposed* $a_n \neq 0$. *Polynomials are multiplied, added and subtracted in the ordinary way.*

We see that for two polynomials, $\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x))$. Thus the Euclidean algorithm works for polynomials:

Lemma 1. *The Euclidean Algorithm: Given two polynomials* $f(x)$, $g(x)$. *Then there exists polynomials* $h(x)$, $r(x)$ *such that* $f(x) = h(x)g(x) + r(x)$ *where* $\deg(r(x)) < \deg(g(x))$

Assume now that $f(x)$ is a polynomial and let $a \in \mathbb{C}$. Then by lemma 1 with $f(x)$, $g(x) = (x - a)$ we can write $f(x) = h(x)(x - a) + r$ where $r = r(x)$ has degree 0 so that it must be a just a number. We see that if $f(a) = 0$, then $0 = f(a) = h(a)(a - a) + r = r$:

Lemma 2. *If $f(x)$ is a polynomial, and if $f(a) = 0$, then $f(x)$ is divisible by $x - a$, that is*

$$f(x) = h(x)(x - a)$$

for some polynomial $h(x)$. The students probably like the language "the division goes up".

Corollary 1. *A polynomial $f(x)$ of degree n can have at most n complex roots.*

(And to all of you math-teachers at this level, you see how this can be exploited to partial fractions, and it is hard to hold back the conjugated pairs...)

Ok, all this was to prove the fundamental theorem of algebra. This main result is built on a few (more) results.

Definition 3. *A function $f(z) : \mathbb{C} \to \mathbb{C}$ is* analytic *if it can be expressed as converging power series in any point $z = a \in \mathbb{C}$. That is*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n.$$

Proposition 1. *Cauchy's Estimate: The coefficients a_n in definition 3 satisfies*

$$|a_n| \leq M(r) \cdot r^{-n}, \ n = 0, 1, 2, \dots$$

where $M(r) = \max_{|z-a|=r} |f(z)|$, and $0 < r$.

Proposition 2. *Liouville's Proposition: If f is analytic and limited in $\mathbb{C}$, then f is constant.*

Proof. Because f is analytic in $\mathbb{C}$, we can let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

with

$$|a_n| \leq M(r) \cdot r^{-n}, \ n = 0, 1, 2, \dots$$

for arbitrary r. As f is limited, there exists K with $M(r) \leq K$ for all r, and so

$$|a_n| \leq K r^{-n}, \text{ for all } r > 0, \ n = 0, 1, 2, \dots.$$

When $r \to \infty$, $a_n \to 0$ for $n \geq 1$ and so $f(z) = a_0$ which is constant. □

Proposition 3. *For a polynomial*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots a_{n-1} z^{n-1} + a_n z^n, \ a_n \neq 0, \ n \geq 1,$$

there exists $m > 0$ *and* $R > 0$ *such that*

$$|f(z)| \geq m|z|^n, \ |z| \geq R.$$

Proof.

$$f(z) = z^n(a_n + a_{n-1}z^{-1} + \dots + a_1 z^{-n+1} + a_0 z^{-n}) \Rightarrow$$
$$|f(z)| \geq |z|^n[(|a_n| - \frac{1}{|z|}(|a_{n-1}| + \dots + |a_1||z|^{-n+2} + |a_0||z|^{-n+1})]$$

Because $\frac{1}{|z|}(|a_{n-1}| + \dots + |a_1||z|^{-n+2} + |a_0||z|^{-n+1}) \to 0$ when $|z| \to \pm\infty$, there exists $R > 0$ such that this term is limited to $\frac{1}{2}|a_n|$ for $|z| > R$ and so

$$|f(z)| \geq \frac{1}{2}|a_n||z|^n, \ |z| \geq R.$$

□

Theorem 1. *The fundamental Theorem of Algebra: A polynomial $f(x)$ with complex coefficients has exactly n complex roots, counted with multiplicity*

Proof. Assume for contradiction that $f(x)$ has no roots. Then the function $g(z) = \frac{1}{f(z)}$ is analytic in the complex plane. By lemma 3 there exists R such that $|f(z)| \geq 1$ for $|z| \geq R$, and then $|g(z)| \leq 1$ for $|z| \geq R$. We have that $g(z)$ is continuous because it is analytic, and then it is compact in the closed set $|z| \leq R$, so bounded. (The image of a compact set by a continuous function is compact). Thus $g(z)$ is analytic and bounded in $\mathbb{C}$. But then, by Proposition 2, g is constant, implying that f is constant. This is a contradiction, and the theorem is proved, taking in lemma 2. □

The fundamental theorem of algebra is the main character in the story, but we will need a few results from linear algebra too. In the following, a vector space is a vector space over $\mathbb{C}$ or $\mathbb{R}$.

Lemma 3. *Let*

$$V \xrightarrow{U} V \xrightarrow{T} V$$

be linear endomorphisms of a not necessarily finite dimensional vector space V. Assume that $\dim\ker(T) = m, \ \dim\ker(U) = n, \ m, n < \infty$, *and that U is surjective. Then*

$$\dim\ker(TU) = m + n.$$

Proof. Choose bases $\ker(T) = \langle t_1, \dots, t_m \rangle$ and $\ker(U) = \langle u_1, \dots, u_n \rangle$. Then $v \in \ker(TU) \Rightarrow U(v) \in \ker(T) \Rightarrow U(v) = \alpha_1 t_1 + \dots + \alpha_m t_m$. Because U is surjective, we can write

$$U(v) = \alpha_1 t_1 + \dots + \alpha_m t_m = \alpha_1 U(\tilde{u}_1) + \dots + \alpha_m U(\tilde{u}_m)$$

with $\tilde{u}_i \in V$. Then

$$v - (\alpha_1 \tilde{u}_1 + \cdots + \alpha_m \tilde{u}_m) = \beta_1 u_1 + \cdots + \beta_n u_n$$

because the left hand side is in $\ker(U)$, so

$$v = \alpha_1 \tilde{u}_1 + \cdots + \alpha_m \tilde{u}_m + \beta_1 u_1 + \cdots + \beta_n u_n.$$

This shows that $\dim \ker(TU) \leq m + n$. Assume

$$0 = \alpha_1 \tilde{u}_1 + \cdots + \alpha_m \tilde{u}_m + \beta_1 u_1 + \cdots + \beta_n u_n.$$

Applying U to both sides, we get

$$\begin{aligned} 0 &= \alpha_1 t_1 + \cdots + \alpha_m t_m \Rightarrow \alpha_1 = \cdots = \alpha_m = 0 \\ &\Rightarrow \beta_1 = \cdots = \beta_m = 0 \end{aligned}$$

proving the lemma. □

Our next character to be introduced, is the n'th order homogeneous differential equation with constant coefficients:

$$\begin{gathered} y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \\ \Updownarrow \\ (D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 I)(y) = 0 \end{gathered}$$

where D is the linear operator $D = \frac{d}{dt}$.

Even if we are interested in the case with real coefficients, we need to assume complex coefficients at first. The following computations give an example of the need of complex numbers in the computation.

Let $y : \mathbb{R} \to \mathbb{C}$ be a solution to the differential equation

$$(D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 I)(y) = 0 \tag{1}$$

By induction, a solution must be infinitely many times differentiable, that is $y \in C^\infty(\mathbb{R}, \mathbb{C})$, and $C^\infty(\mathbb{R}, \mathbb{C})$ is an infinite dimensional vector space.

Lemma 4. *For $c \in \mathbb{C}$, the solutions to $y' = cy$ are the functions $y(t) = re^{ct}$ for $r \in \mathbb{C}$.*

Proof. $(re^{ct})' = cre^{ct}$ so re^{ct} satisfies the equation. Conversely, if $y' = cy$ then

$$\left(\frac{y}{e^{ct}}\right)' = \frac{y' e^{ct} - y(e^{ct})'}{(e^{ct})^2} = \frac{cye^{ct} - yce^{ct}}{e^{2ct}} = 0$$

so that $\frac{y}{e^{ct}} = r$ constant, giving $y = re^{ct}$ □

Lemma 5. *For $c \in \mathbb{C}$ the operator $D - cI$ is surjective.*

Proof. If $c = 0$ this is obvious because any any $y \in C^\infty(\mathbb{R}, \mathbb{C})$ is integrable. Assume $c \neq 0$. Then

$$\begin{gathered} y' - cy = f \\ \Updownarrow \\ e^{-ct}y' - ce^{-ct}y = e^{-ct}f \\ \Updownarrow \\ (e^{-ct}y)' = e^{-ct}f \\ \Updownarrow \\ e^{-ct}y = u \text{ for some } u \in C^\infty(\mathbb{R}, \mathbb{C}) \\ \Updownarrow \\ y = e^{ct}u. \end{gathered}$$

□

4.2 The Fairytale

Now, we answer the questions 5 and 6 from the students:

Theorem 2. *Let $p(t)$ be a polynomial with complex coefficients of degree $n \geq 1$ and let $p(D)$ be the corresponding n-th order linear differential operator. The solution space to $p(D) = 0$ in $C^\infty(\mathbb{R}, \mathbb{C})$ is n-dimensional, or equivalently* $\ker(p(D))$ *has dimension n.*

Proof. The proof will be by induction on the degree of p. If $\deg p = 1$, the result follows from lemma 4. Assume the result true for $degp = n$. By the fundamental theorem of algebra,

$$p(t) = (t - c_1)) \cdots (t - c_n)(t - c_{n+1}) = q(t)(t - c_{n+1})$$

so that

$$p(D) = q(D)(D - c_{n+1}I).$$

By the induction hypothesis, $\dim \ker p(D) = n$ and by the induction start $\dim \ker(D - c_{n+1}I) = 1$. Furthermore, we have the sequence of operators

$$C^\infty(\mathbb{R}, \mathbb{C}) \overset{(D-c_{n+1}I)}{\longrightarrow} C^\infty(\mathbb{R}, \mathbb{C}) \overset{q(D)}{\longrightarrow} C^\infty(\mathbb{R}, \mathbb{C})$$

which satisfies the assumptions of lemma 5, and so

$$\dim_{\mathbb{C}} \ker(p(D)) = n + 1$$

and the theorem is proved by induction. □

I will not continue as strictly as before. I just close the fairytale with the following result, which I think anyone of you can solve with the basis already given:

Theorem 3. *Let $p(t)$ be a polynomial with real coefficients of degree $n \geq 1$ and let $p(D)$ be the corresponding n-th order linear differential operator with constant coefficients. The solution space to $p(D)(y) = 0$ in $C^\infty(\mathbb{R})$ is n-dimensional.*

5 Epilog

Well...

1. Does my background in mathematics make me an enthusiastic lecturer?

2. Is the deeper knowledge necessary for a lecturer, or just nice?

3. Is the mathematical knowledge a disadvantage for teaching the engineering students? That is, are the toolmakers unable to teach using the tools?

4. Why are there no questions about didactics? (Just give it to me!)

Of course, you all knew all this from before. But the main question is if you would be a good math teacher if you didn't!