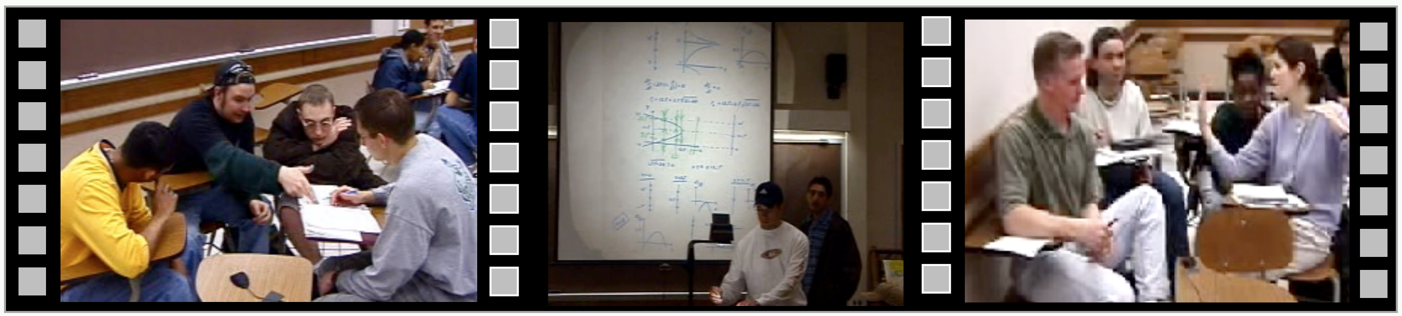
The Inquiry Oriented Differential Equations Project: Addressing Challenges Facing Undergraduate Mathematics Education



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MatRIC Modelling Colloquium
August 2016

MatRIC Centre for Research, Innovation and Coordination of Mathematics Teaching

Overview of talk

- Challenges and opportunities in undergraduate mathematics education
- An overview of the Inquiry oriented differential equations project
- Two unexpected examples of what is possible

A challenge facing undergraduate mathematics education

In addition to difficulties in whistling ….

- Significant difficulty in conceptualizing the limit processes underlying the notions of derivative and integral (Orton, 1980)
- Discrepancy between the formal definitions students were able to quote and the criteria they used check properties such as functions, continuity, derivative (Tall & Vinner, 1981)
- Students’ difficulties with logical reasoning and proofs (Alibert & Thomas, 1991; Schoenfeld, 1985; Selden & Selden, 1981)
- Students’ difficulties in connecting graphs with physical concepts and the real world (McDermott, Rosenquist, & Van Zea, 1987; Rasmussen, 2001; Svec, 1995)
- Students’ difficulties in learning the basic notions of linear algebra and differential equations (Harel, 1989; Rasmussen, 2001)

Other Challenges (and Opportunities) in Undergraduate Mathematics Education

- Increasing number of students (with diverse cultural and academic backgrounds) entering university
- Decline in the number of mathematics and science majors
- Poor achievement in foundational courses such as calculus
- Need for prospective teachers to experience innovative teaching as learners of mathematics and science
- Opportunities for compatible rethinking of secondary and tertiary mathematics
- More occasions for collaborations between mathematicians and undergraduate math education researchers

These challenges and opportunities point to the need to explore innovative approaches to teaching and learning.

An Inquiry-Oriented Approach

Students investigate challenging problems

A - Learn new mathematics by engaging in mathematical activity

B - Affect beliefs about themselves and about the nature of mathematics and the nature of school learning

Teacher investigates student thinking

A - Build models of student thinking

B - Learn new mathematics

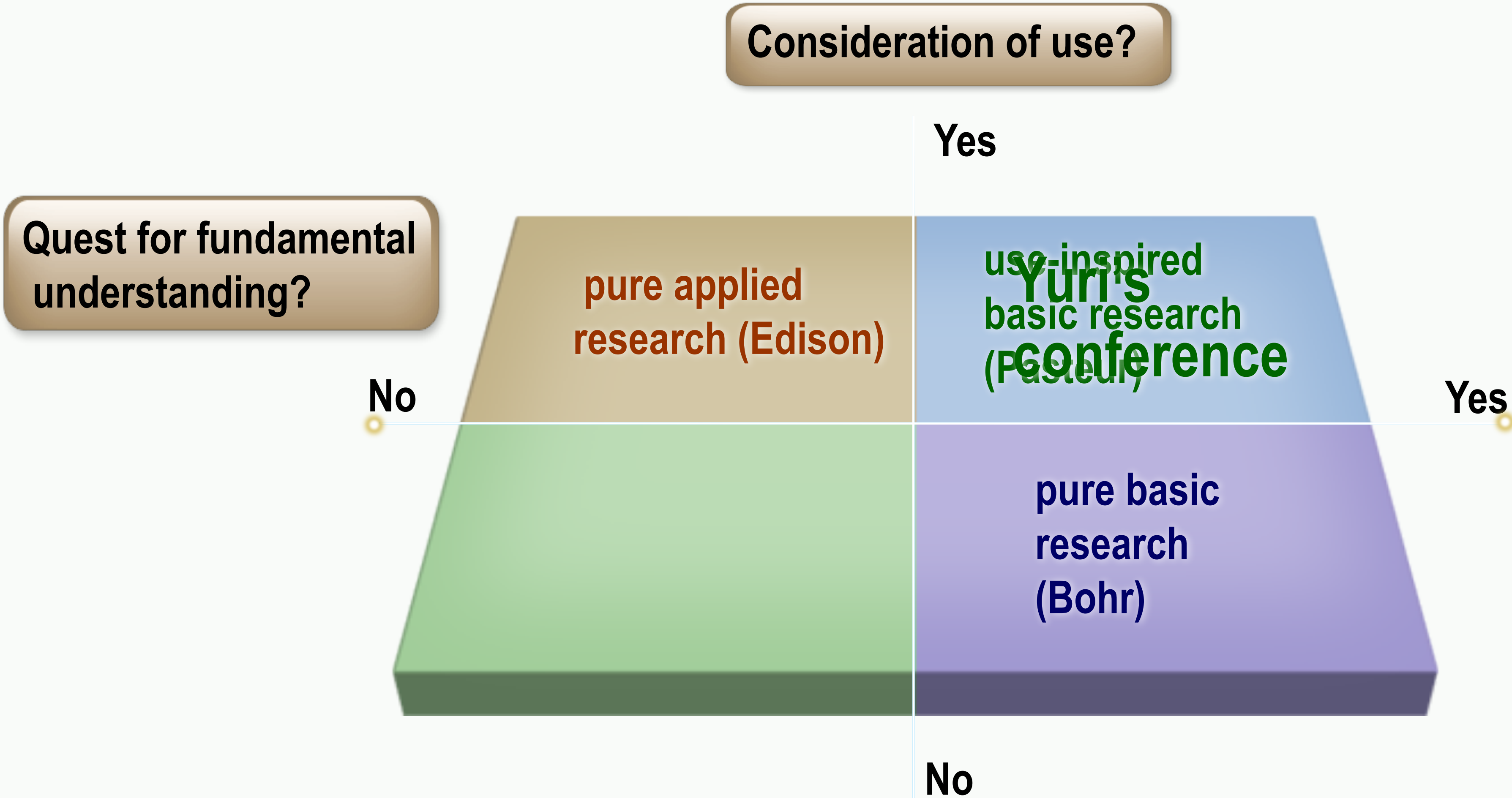
C - Figure out what next question or task to pose

(Rasmussen & Kwon, 2007)

Why Differential Equations?

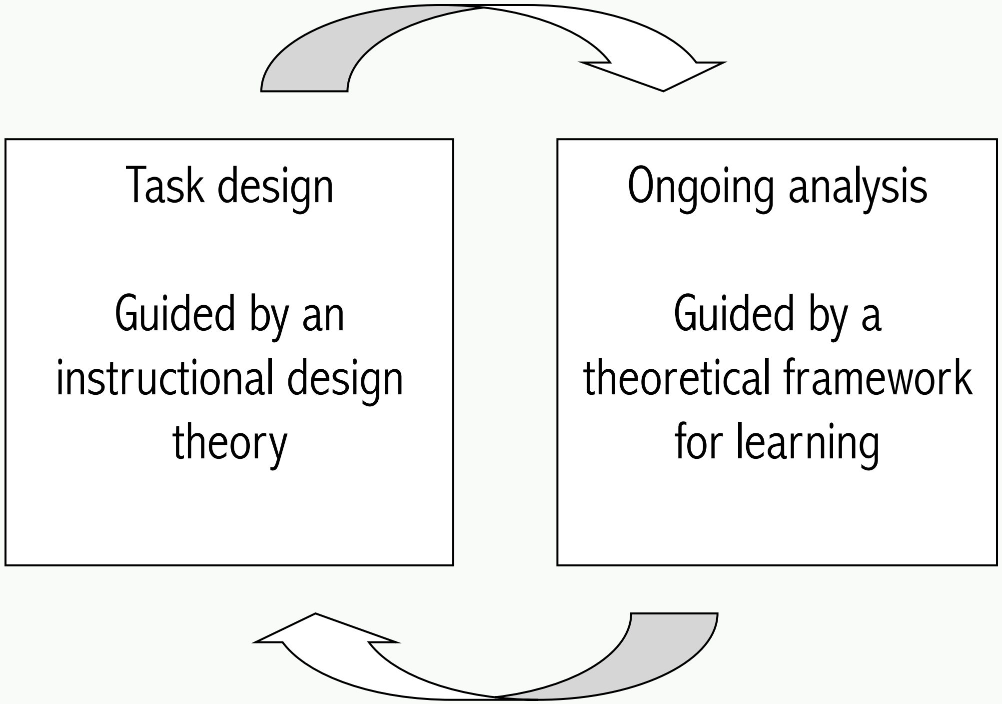
1. An important course a variety of majors
2. Little educational research done in this area
3. Traditional approach-organized primarily around analytic methods
4. Modern approach-organized around modeling, graphical, numerical and qualitative methods
5. Strong connections with calculus and linear algebra

Research Inspired by:



Quadrant model of Scientific Research (Stokes, 1997)

Design Based Research (Cobb, 2000)



Systematic investigations of student learning

Quantitative methods (Statistical analysis)	Qualitative methods (Interpretative analysis)

- Individual interviews
  - to probe student understanding in depth
- Written questions (pretests and post-tests)
  - to ascertain prevalence of specific difficulties
  - to assess effectiveness of instruction
- Descriptive and explanatory studies of classroom interactions
  - to provide insights to guide instructional design
  - to examine mechanisms that underlie classroom learning

Inquiry-Oriented Differential Equations

Background theory

- Dynamical systems point of view
- Instructional design theory of Realistic Mathematics Education

- Co-emergent analytic, numerical, and graphical approaches
- Systematic study of student thinking
- Modeling

Innovative curriculum

- Java applets

Innovative technological tools

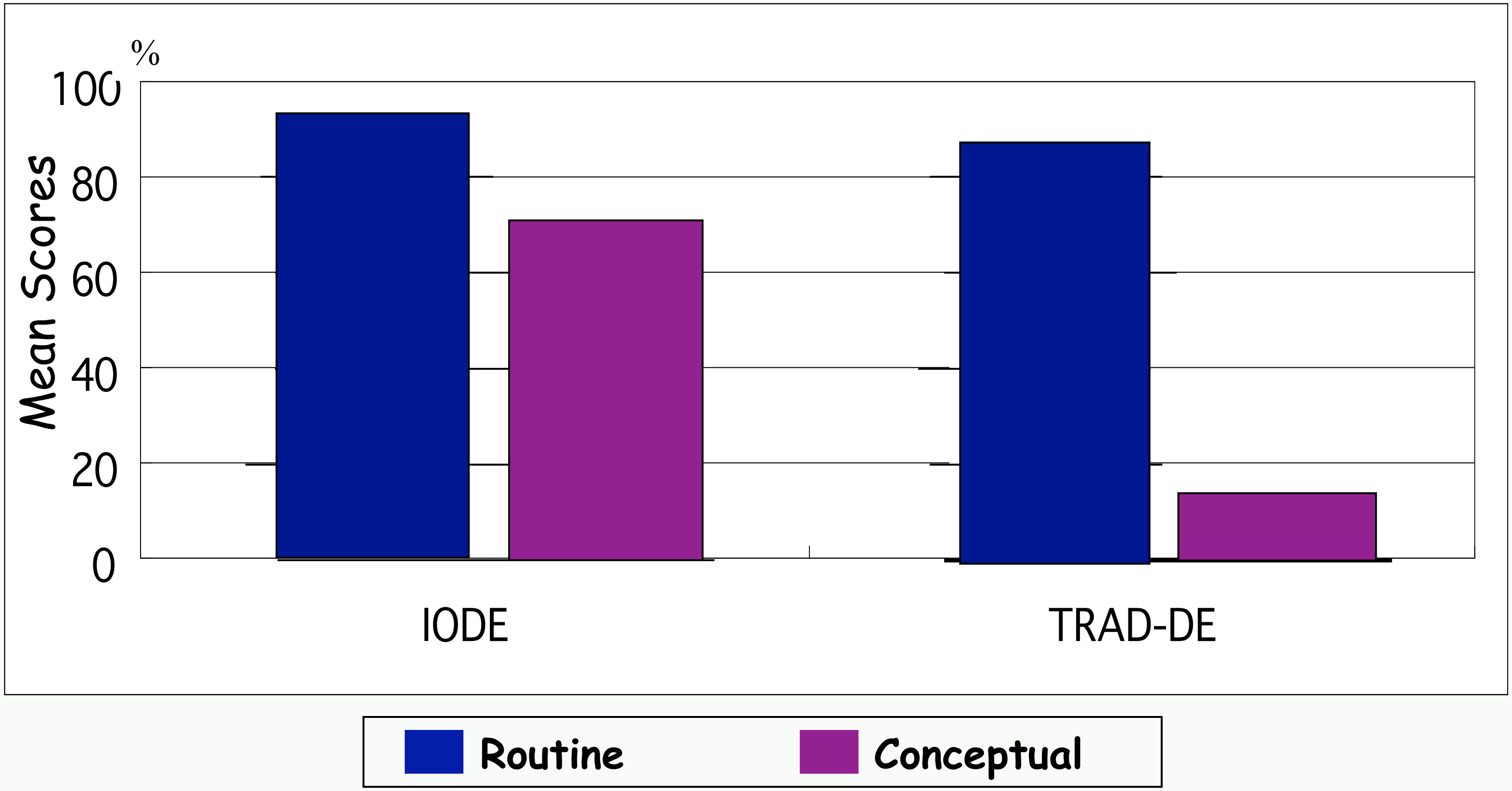
- Social Negotiation of Meaning
- Role of the teacher

Innovative pedagogy

In the IO-DE project

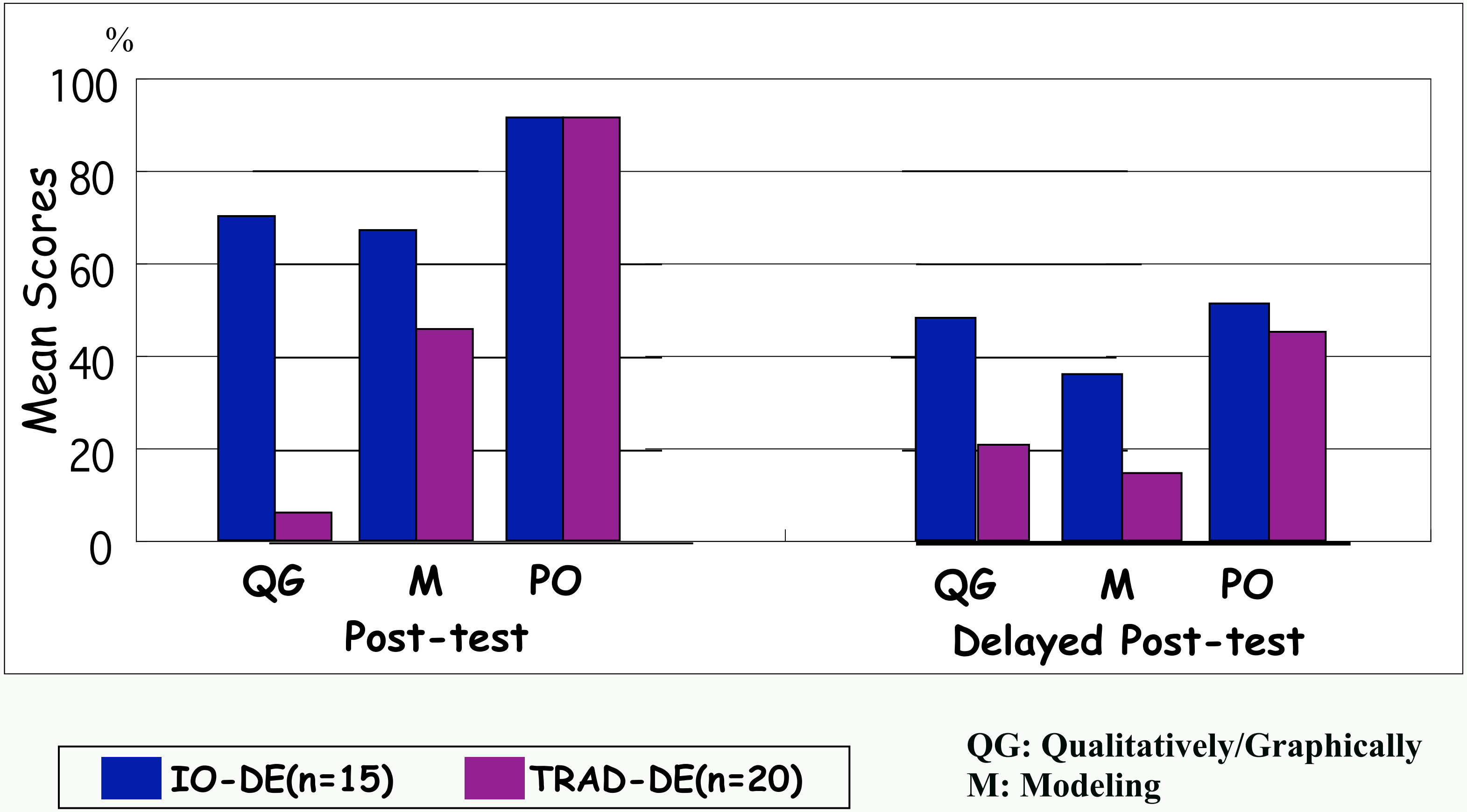
- Students (with guidance) reinvent many key mathematical ideas
- Guided reinvention does take more time
- So what are the benefits for learners? Do they learn ideas more deeply? Do they retain their knowledge longer? Do they develop more productive dispositions about mathematics, about themselves, and their ability to do mathematics?

Comparison Study
Assessment of student learning



- 4 different sites, N = 111

Students' retention of mathematical knowledge and skills in differential equation



QG: Qualitatively/Graphically
M: Modeling
PO: Procedurally Oriented

IO-DE effect on student attitudes:

A student testimony

When I started taking classes in college after learning only how to do simple calculations in high school math, I totally couldn’t follow the in-depth proofs my college professors were explaining. I was so frustrated many times. I started questioning my math ability and began to think that majoring in mathematics education might be not the path I’m supposed to take. However, I decided to stick with math ed, figuring I just needed to master high school math in order to be a high school teacher.

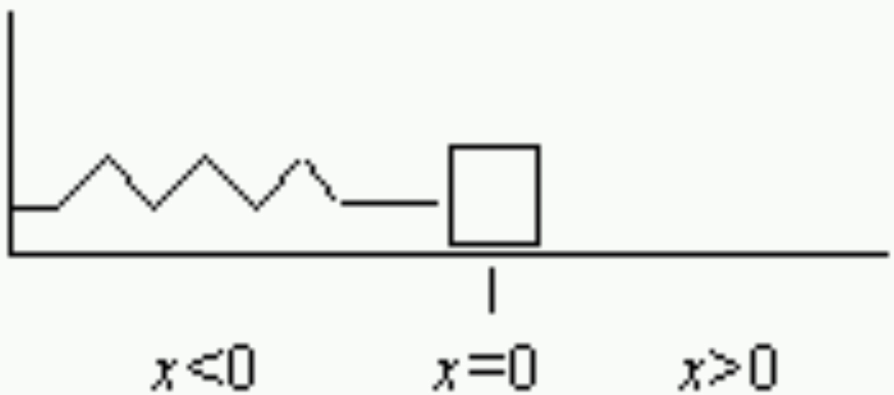
What gave me an opportunity to gain my confidence back was differential equations class. Through differential equations, I experienced what it really meant to do math, and I regained some of my lost confidence. At the least, I felt like I escaped from the pit of my academic slump and got some results to prove it….

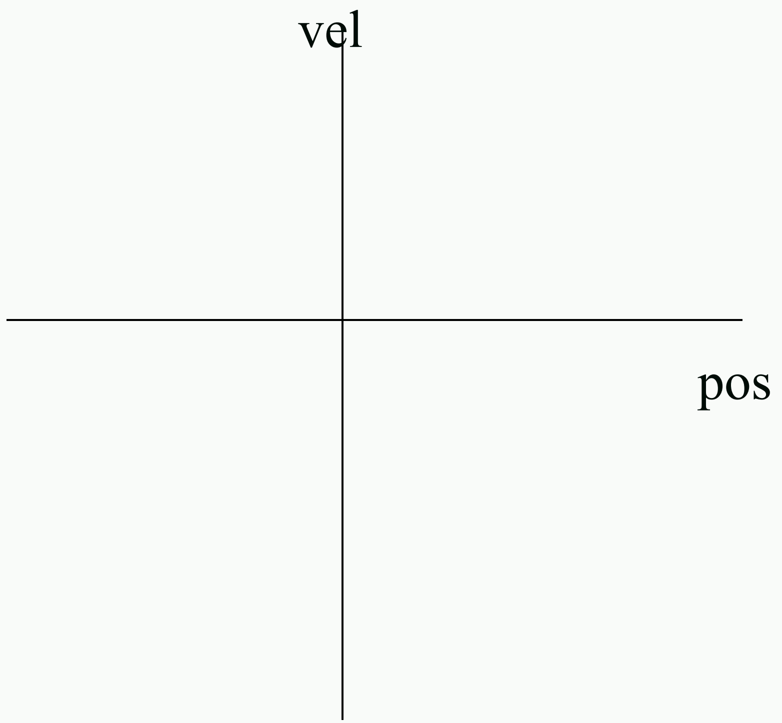
Example 1: Reinvention of Eigensolutions

- Innovative Approach to Systems
- Grounded in the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999) and inquiry-oriented classroom learning environments – Emergent Model heuristic
- Result of several years of classroom based research
- Reinvention process involves a reversal of the standard analytic approach

Prediction Task

Imagine that I pull the mass back and release it. How do you think the motion of the mass might be represented in the position-velocity plane? Please create various position versus velocity graphs for different amounts of friction.





Exploration Task

After developing the differential equations for this situation, students use a computer program to empirically explore changes to the vector field as the friction coefficient for the spring mass situation varies.

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \quad \Longrightarrow \quad \begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2x - by \end{aligned}$$

where x represents the position of the spring mass relative to the equilibrium point, y represents the velocity of the mass, b is the coefficient of friction, spring constant k set to 2, and value of mass m set to 1.

Exploration with technology

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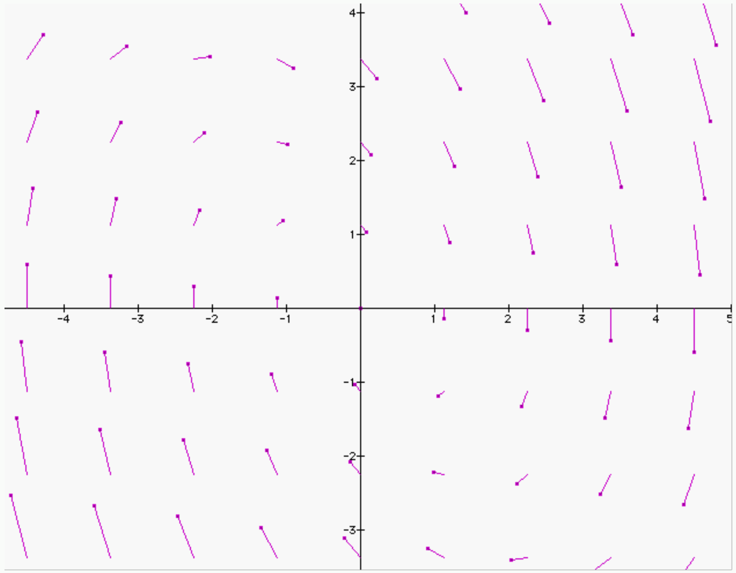

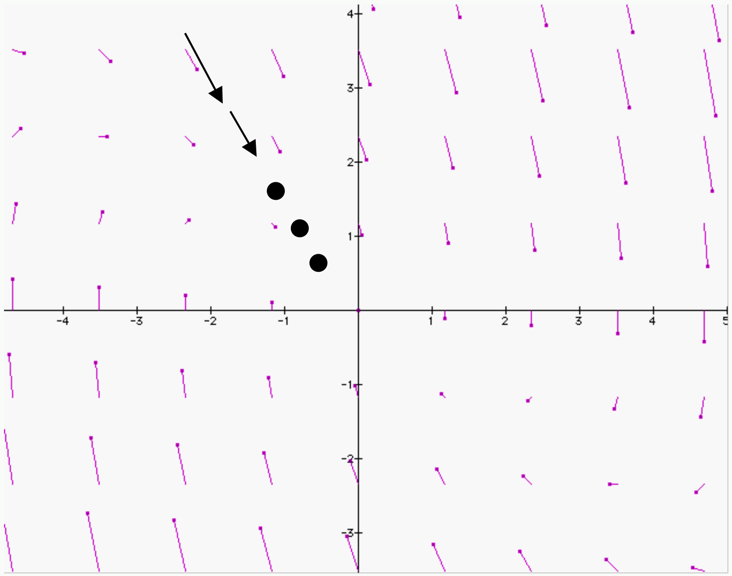

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Spiraling and non-spiraling solutions

Because students did not predict the straight line in the position-velocity plane, they become intellectually interested in determining the slope of this straight line. How would you algebraically determine the slope of the straight line of vectors?

Mathematization Task

Students model their empirical observations by inventing an algebraic approach for determining the slope of the line(s) along which vectors point directly toward the origin. Students then figure out the equations for the pair of functions that solve the system of differential equations along this line.

Mathematization

Student reinvention capitalizes on their knowledge that a line going through the origin is of the form $y=mx$ and that the slope of the vectors is the ratio of the two rate of change equations. Thus, m = y/x and

$$m = \frac{dy/dt}{dx/dt} = \frac{cx + dy}{ax + by}$$

Equating the two expressions for the slope yields

$$\frac{y}{x} = \frac{cx + dy}{ax + by}$$

Replacing y with mx and simplifying yields a quadratic equation in m. Thus, depending on the discriminant, there are two, one, or no real values for the slope.

Mathematatization (continued)

Substituting $y = mx$ into $\frac{dx}{dt} = ax + by$ yields $\frac{dx}{dt} = rx$.

Using separation of variables yields $x(t) = k_1 e^{rt}$.

Similarly, the same method can be used to solve $\frac{dy}{dt} = cx + dy$ for $y(t)$.

The process can be repeated when there is a second SLS and the general solution is formed based on the fact that the differential equations are linear and homogeneous. Students' self invented method works equally well for complex values of the slope m.

General Activity

Students generalize their approach by determining the general solution for this situation and other situations involving real and complex eigenvalues.

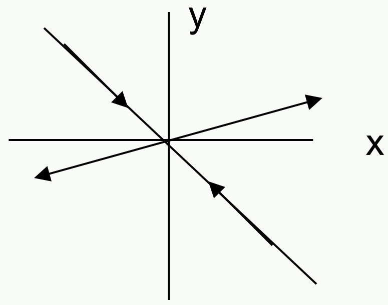
Looking Back at Student Reinvention

Reversal of the standard analytic approach

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

- Standard approach - find eigenvalues first and then eigenvectors

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

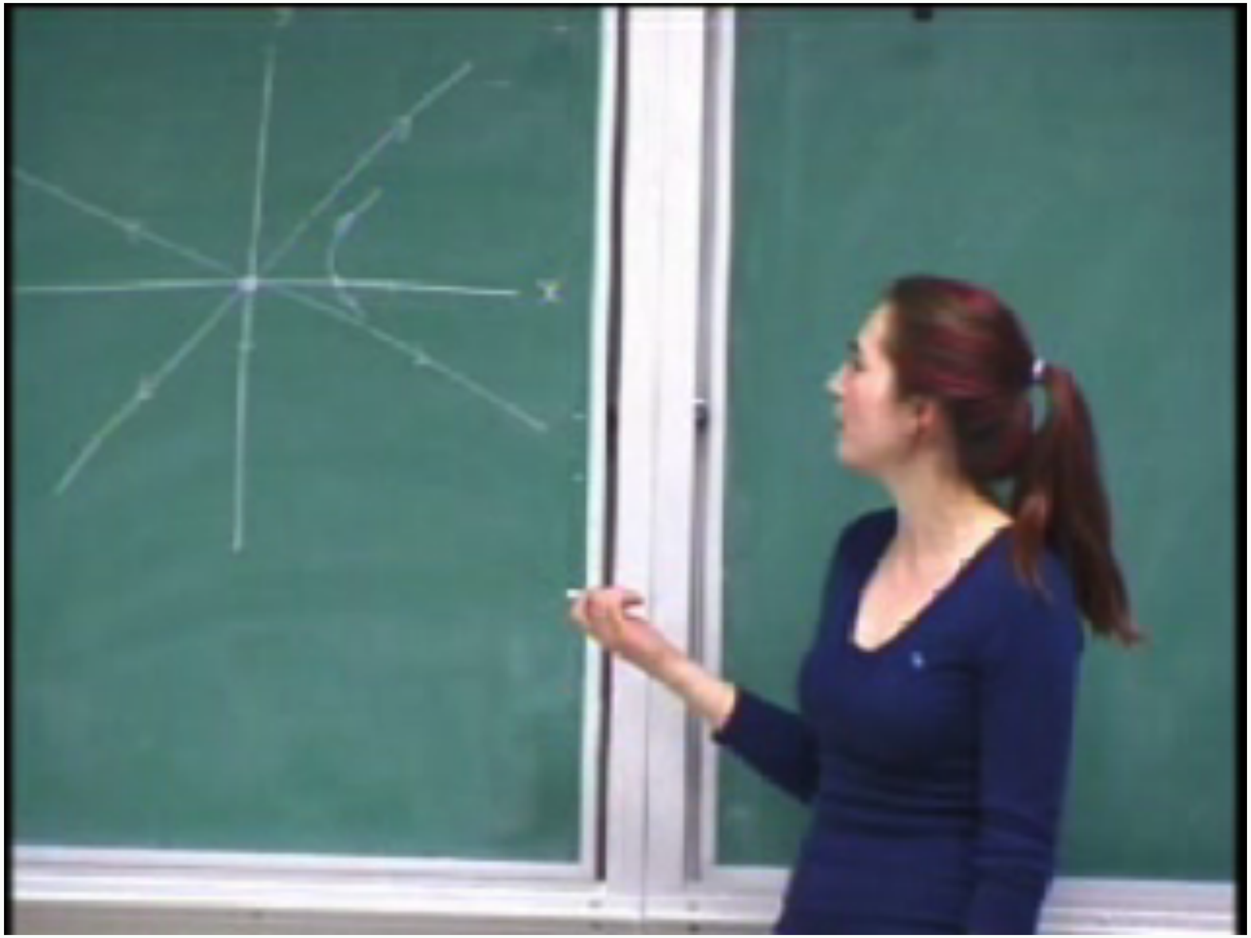


- What is more conceptually coherent for learners is to actually find eigenvectors first (or rather the slope of the line of eigenvectors) and then eigenvalues

The equations for the SLSs subsequently become a *tool-for* reasoning about the shapes of the graphs for all other solutions

Formal Activity: Using graphs of SLSs as a tool for reasoning

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



Anna: I sketch a graph of um e^t and e^{4t} to kind of figure out… If t is negative, well I was looking at this and I realized that since t is negative, then both values are approaching zero [the graphs of the exponential functions], and therefore this [u-shaped graph] would not be correct because both values, x is getting positive and y is getting negative but a bigger negative. So, um, if this [graphs of e^t and e^{4t}] both approaches zero, then um both equations x(t) and y(t) also have to approach zero.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Reasoning with the components of the general solution
- Graphs of e^t and e^{4t} as tools for reasoning
- Explicitly attention to time (in contrast to previous argument that resulted in the u-shaped graph)

Using ratio of y(t)/x(t) as a tool for reasoning

$$\lim_{x \to \infty} \frac{y(t)}{x(t)} = 1 \qquad \lim_{x \to -\infty} \frac{y(t)}{x(t)} = -\frac{1}{2}$$

Anna: From that, I realized that um no matter where we start, where our initial condition is, if t approaches a positive infinity, then, um the graph tries to look, I'll rephrase it, the graph does not try to look, the um, the slope of the graph, because this [y(t)/x(t)] represents the slope, will look more like, or will get closer to 1, and if t approaches negative infinity, then the slope of the graph will look closer to –1/2.

- Conclusions apply to any initial condition
- Builds from previous argument
- Careful attention to language

Flow of the instructional sequence

(Prediction, Exploration, Mathematization, Generalization)

In this example we saw both the reinvention of a significant mathematical idea and then subsequent use of this idea to reason and make warranted conclusions.

Realistic Mathematics Education Emergent Model Heuristic
Situational activity involves students working toward mathematical goals in an experientially real setting.
Referential activity involves models-of that refer (implicitly or explicitly) to physical and mental activity in the original task setting.
General activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting.
Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity.

Example 2: Reinvention of a bifurcation diagram

Fish Harvesting Task: $\frac{dP}{dt} = 2p(1 - \frac{p}{25})$

A scientist at a fish hatchery has previously demonstrated that the above rate of change equation is a reasonable model for predicting the number of fish that the hatchery can expect to find in their pond.

Recently, the hatchery was bought out by fish.com and the new owners are planning to allow the public to catch fish at the hatchery (for a fee of course). The new owners need to decide how many fish per year they should allow to be harvested. Prepare a report for the new owners that illustrate the implications that various choices for harvesting will have on future fish populations.

Task presented students with an opportunity to engage in:

- Modeling
- Symbolizing

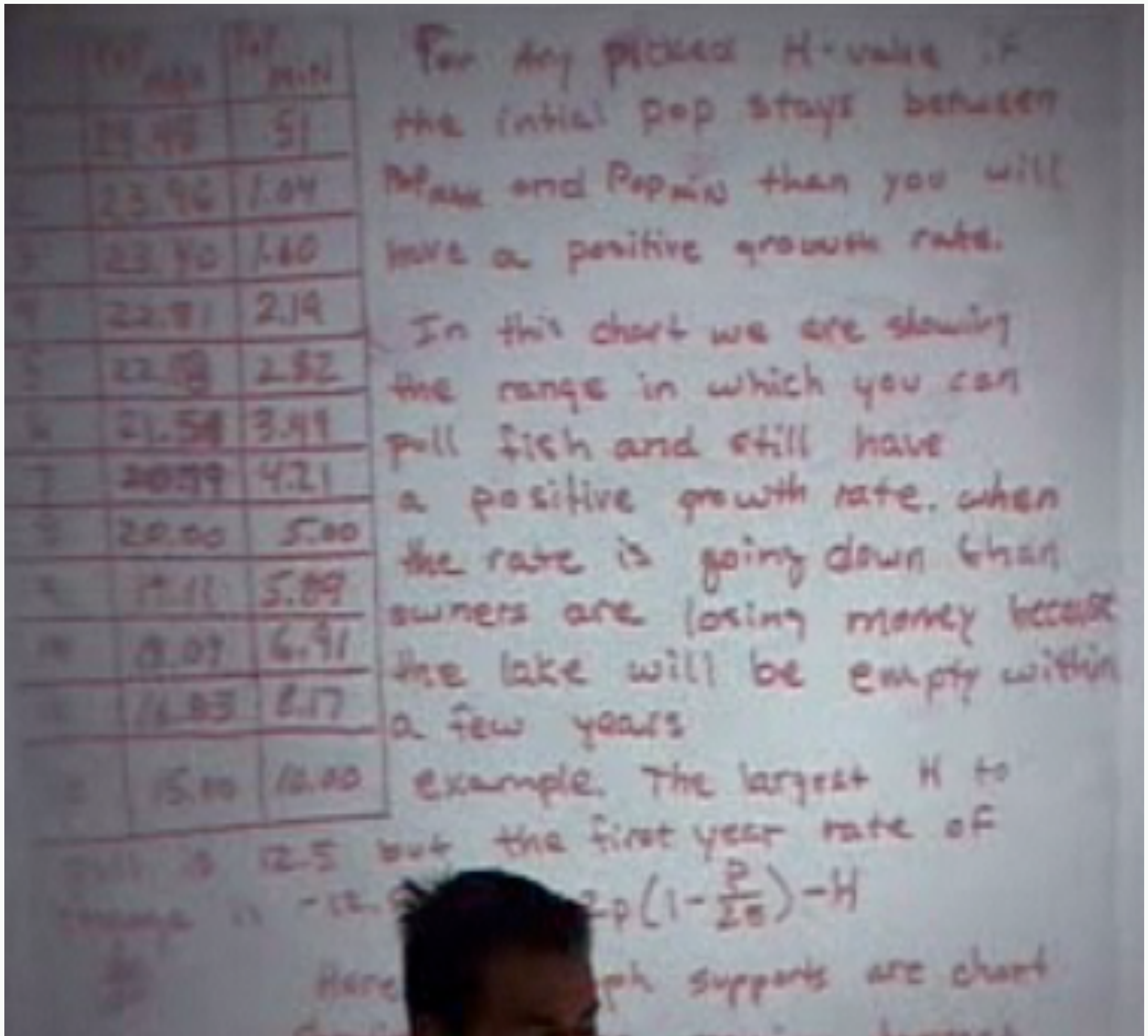
→ Reinvention of a bifurcation diagram

$$\frac{dP}{dt} = 2p(1 - \frac{p}{25}) \longrightarrow \frac{dP}{dt} = 2p(1 - \frac{p}{25}) - k$$

Key features of the task:

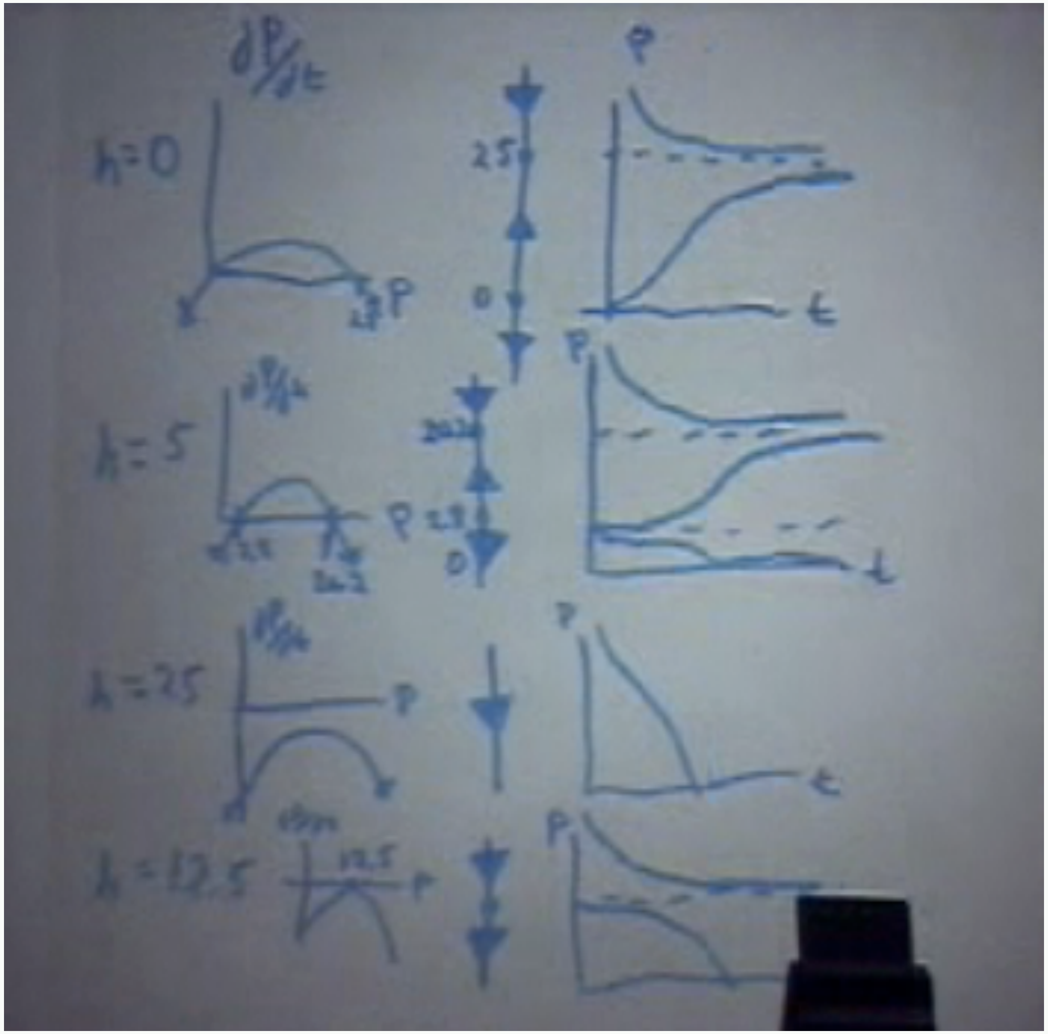
- Did not ask students to find the best harvesting rate
- Limited the report to one page

Group 1 Presentation Slide

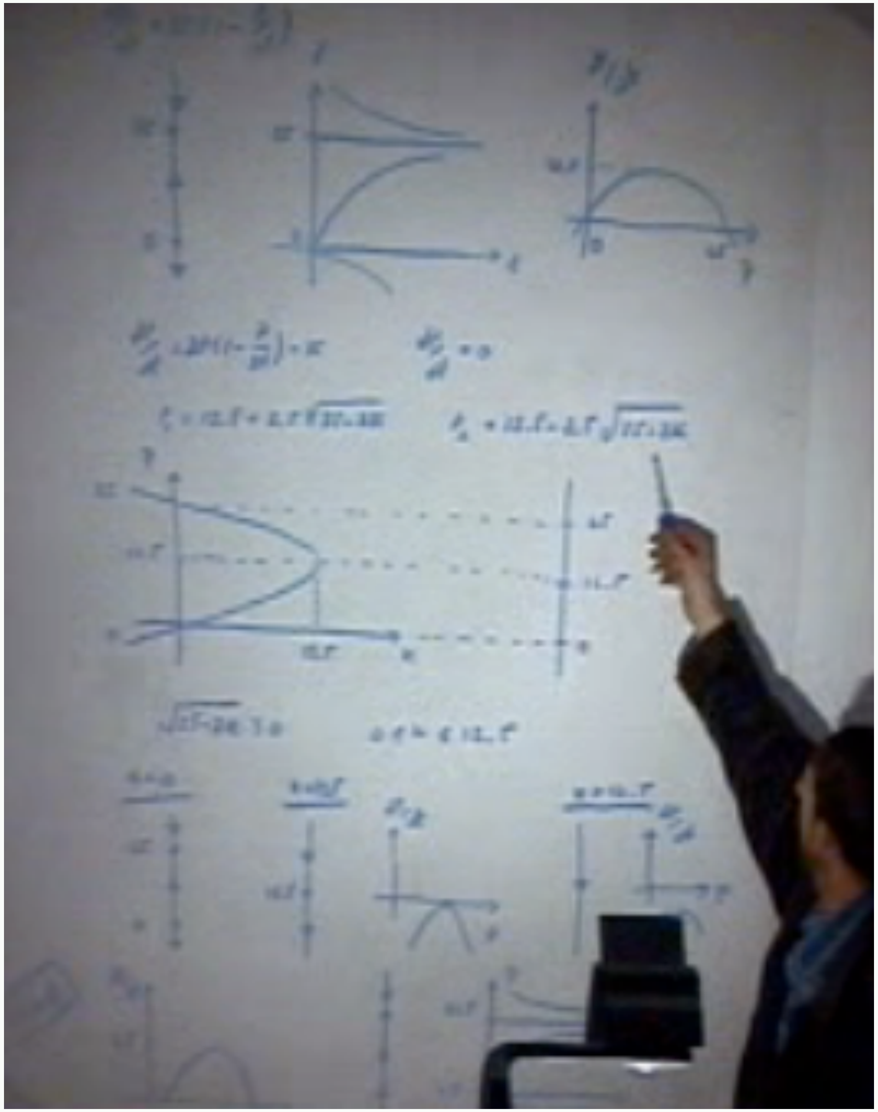


Group 2 Presentation Slide

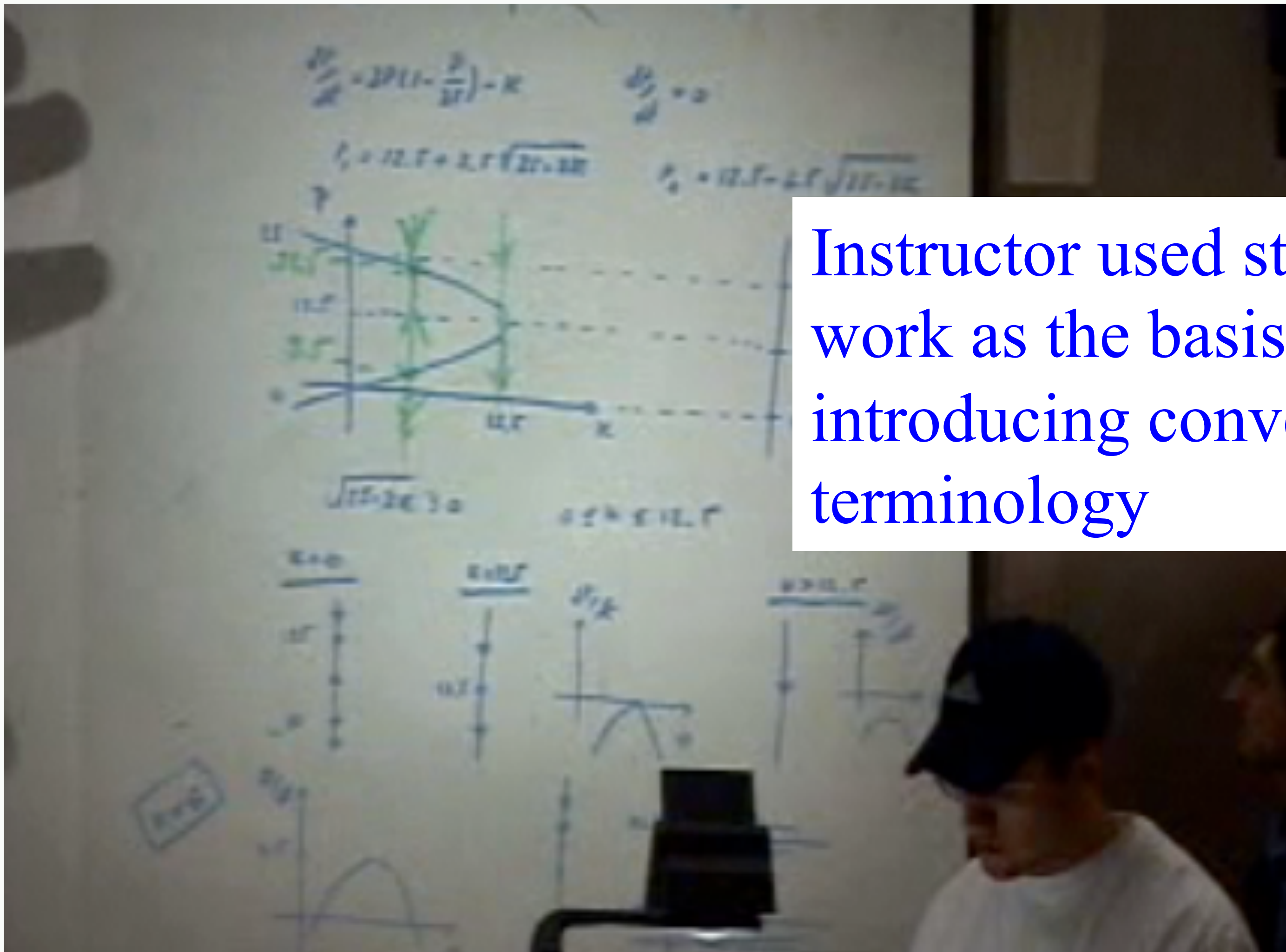
Reasoning with rate of change as a dynamic object from which you can make justified inferences about the structure of the solution space



Group 3 Presentation Slide



Group 4 Presentation Slide 2 as elaborated during discussion



Instructor used student work as the basis for introducing conventional terminology

A student product during a different semester

Sheets | Charts | SmartArt Graphics | WordArt

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1															
2	2P(1-P/25)														
3															
4	P	dP/dt	1	2	3	4	5	6	7	8	9	10	11	12	12.5
5	1	1.92	0.92	-0.08	-1.08	-2.08	-3.08	-4.08	-5.08	-6.08	-7.08	-8.08	-9.08	-10.08	-10.58
6	2	3.68	2.68	1.68	0.68	-0.32	-1.32	-2.32	-3.32	-4.32	-5.32	-6.32	-7.32	-8.32	-8.82
7	3	5.28	4.28	3.28	2.28	1.28	0.28	-0.72	-1.72	-2.72	-3.72	-4.72	-5.72	-6.72	-7.22
8	4	6.72	5.72	4.72	3.72	2.72	1.72	0.72	-0.28	-1.28	-2.28	-3.28	-4.28	-5.28	-5.78
9	5	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-4.5
10	6	9.12	8.12	7.12	6.12	5.12	4.12	3.12	2.12	1.12	0.12	-0.88	-1.88	-2.88	-3.38
11	7	10.08	9.08	8.08	7.08	6.08	5.08	4.08	3.08	2.08	1.08	0.08	-0.92	-1.92	-2.42
12	8	10.88	9.88	8.88	7.88	6.88	5.88	4.88	3.88	2.88	1.88	0.88	-0.12	-1.12	-1.62
13	9	11.52	10.52	9.52	8.52	7.52	6.52	5.52	4.52	3.52	2.52	1.52	0.52	-0.48	-0.98
14	10	12	11	10	9	8	7	6	5	4	3	2	1	0	-0.5
15	11	12.32	11.32	10.32	9.32	8.32	7.32	6.32	5.32	4.32	3.32	2.32	1.32	0.32	-0.18
16	12	12.48	11.48	10.48	9.48	8.48	7.48	6.48	5.48	4.48	3.48	2.48	1.48	0.48	-0.02
17	13	12.48	11.48	10.48	9.48	8.48	7.48	6.48	5.48	4.48	3.48	2.48	1.48	0.48	-0.02
18	14	12.32	11.32	10.32	9.32	8.32	7.32	6.32	5.32	4.32	3.32	2.32	1.32	0.32	-0.18
19	15	12	11	10	9	8	7	6	5	4	3	2	1	0	-0.5
20	16	11.52	10.52	9.52	8.52	7.52	6.52	5.52	4.52	3.52	2.52	1.52	0.52	-0.48	-0.98
21	17	10.88	9.88	8.88	7.88	6.88	5.88	4.88	3.88	2.88	1.88	0.88	-0.12	-1.12	-1.62
22	18	10.08	9.08	8.08	7.08	6.08	5.08	4.08	3.08	2.08	1.08	0.08	-0.92	-1.92	-2.42
23	19	9.12	8.12	7.12	6.12	5.12	4.12	3.12	2.12	1.12	0.12	-0.88	-1.88	-2.88	-3.38
24	20	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-4.5
25	21	6.72	5.72	4.72	3.72	2.72	1.72	0.72	-0.28	-1.28	-2.28	-3.28	-4.28	-5.28	-5.78
26	22	5.28	4.28	3.28	2.28	1.28	0.28	-0.72	-1.72	-2.72	-3.72	-4.72	-5.72	-6.72	-7.22
27	23	3.68	2.68	1.68	0.68	-0.32	-1.32	-2.32	-3.32	-4.32	-5.32	-6.32	-7.32	-8.32	-8.82
28	24	1.92	0.92	-0.08	-1.08	-2.08	-3.08	-4.08	-5.08	-6.08	-7.08	-8.08	-9.08	-10.08	-10.58
29	25	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-12.5
30	26	-2.08	-3.08	-4.08	-5.08	-6.08	-7.08	-8.08	-9.08	-10.08	-11.08	-12.08	-13.08	-14.08	-14.58
31	27	-4.32	-5.32	-6.32	-7.32	-8.32	-9.32	-10.32	-11.32	-12.32	-13.32	-14.32	-15.32	-16.32	-16.82
32	28	-6.72	-7.72	-8.72	-9.72	-10.72	-11.72	-12.72	-13.72	-14.72	-15.72	-16.72	-17.72	-18.72	-19.22
33	29	-9.28	-10.28	-11.28	-12.28	-13.28	-14.28	-15.28	-16.28	-17.28	-18.28	-19.28	-20.28	-21.28	-21.78
34	30	-12	-13	-14	-15	-16	-17	-18	-19	-20	-21	-22	-23	-24	-24.5
35															
36															
37															
38															

Fish.com task and student reinvention is then used as the basis for further mathematical work without a real world context

For example ……

Reinvention of Bifurcation Diagram

Differs from textbook presentations in two important ways

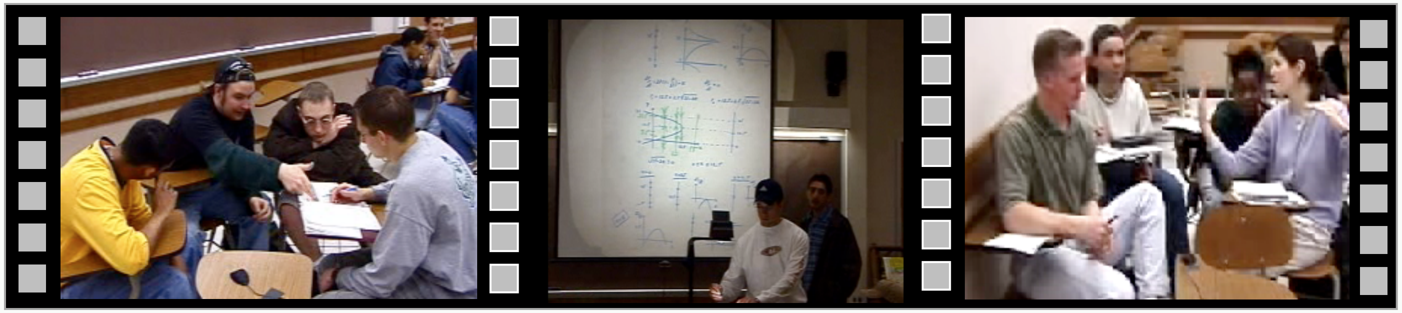
1) Texts typically first define bifurcation and then give a bifurcation diagram to illustrate this idea - the reverse process happens for students
2) One influential reform oriented text instructs students to create a bifurcation diagram by drawing many phase lines and connecting the dots (equilibrium solutions) - for students the curve comes first and then phase lines are used to add additional meaning

Conclusion

The IO-DE is an example of a research-based curriculum that:

- Makes mathematics meaningful to students via emergent modeling
- Develops students’ mathematical reasoning ability and positively affects attitudes and disposition

The End - Thanks for Listening



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More information about the IODE project is available at https://iode.wordpress.ncsu.edu/