Research on Learning and Teaching University Mathematics

Chris Rasmussen

San Diego State University

San Diego State University

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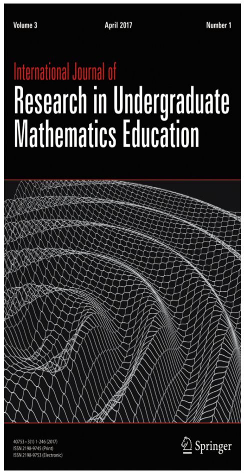
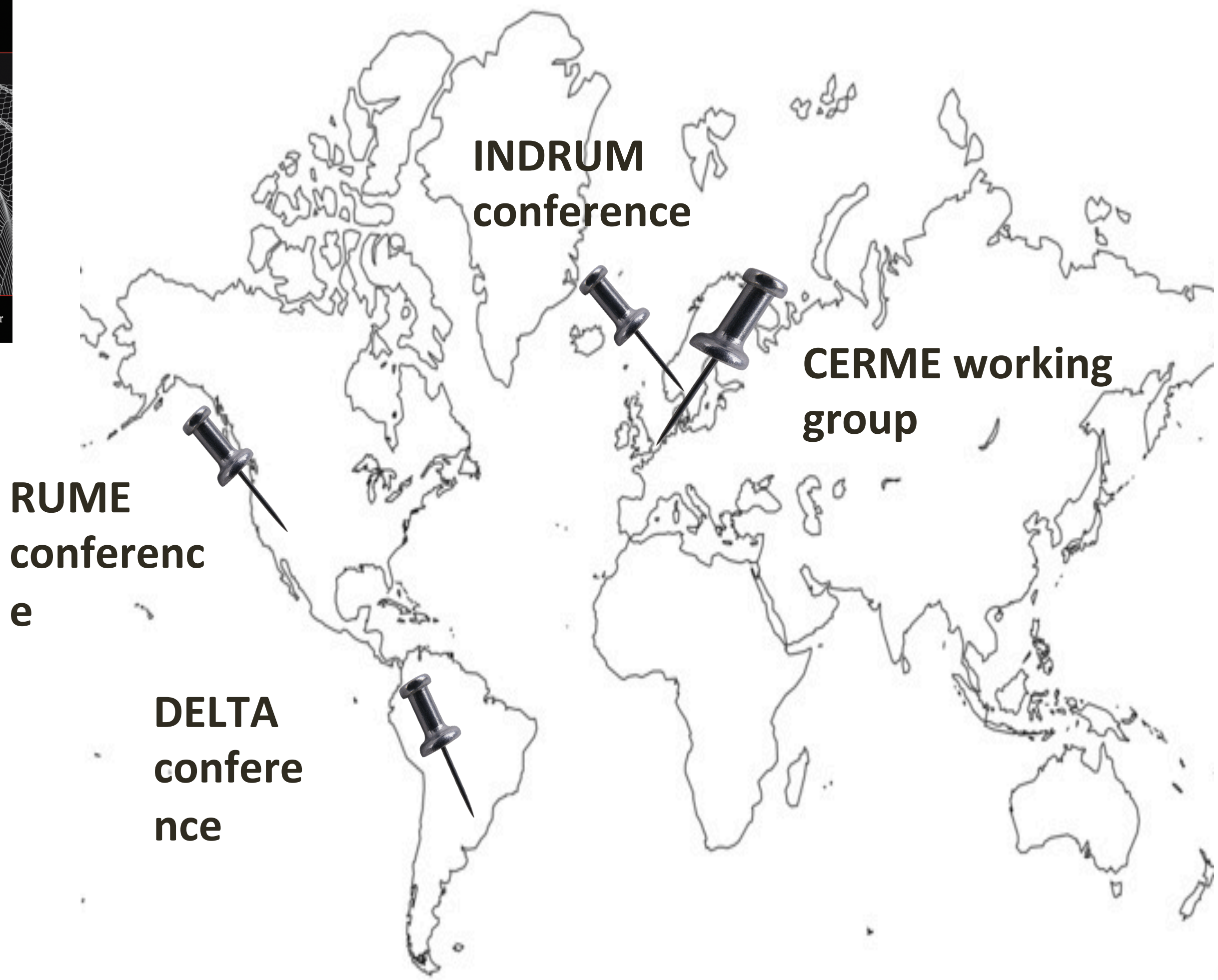
Excellence in Teaching Higher Education Mathematics

MatRIC Centre for Research, Innovation and Coordination of Mathematics Teaching

Growth of Research in Undergraduate Mathematics Education (RUME)





RUME Pattern of Growth

Similar to the elementary and secondary literature, RUME has followed a pattern of

- Identifying and studying student difficulties and cognitive obstacles followed by
- Investigations of the processes by which students learn particular concepts, evolving into
- Classroom studies (or close approximations thereof), including the effects of curricular and pedagogical innovations on student learning, and, more recently
- Research on teacher (including graduate student instructor, lecturers, etc.) knowledge, beliefs, and practices.

A Synthesis of the Post Calculus RUME Literature

- Every 10 years the National Council of Teachers of Mathematics in the US publishes a Handbook that synthesizes the research in mathematics education
- This talk is partly based on the 2017 handbook chapter on post calculus mathematics education research by myself and Megan Wawro
- We reviewed over 200 articles published since 2005
- The chapter is organized in three main sections:
  - Research on student learning of particular content (linear algebra, differential equations, analysis, abstract algebra)
  - Research on Teaching (lecture, inquiry, professional development)
  - Future Directions (theoretical/methodological coordination, mathematical practices, connections to other STEM disciplines)

Outline of Presentation

Part 1 – Research highlights on student learning of linear algebra (as an example of other similar sections)

Part 2 - Research highlights on undergraduate mathematics teaching

Part 3 - Future directions: Connecting RUME and other discipline based educational research

Part 1 - Linear Algebra

- Started with the 2007 handbook - linear algebra research review dominated by *The Teaching and Learning of Linear Algebra,* edited by Dorier (2000). Three themes from this prior work:
    - categorizations for students' reasoning and difficulties
    - discussions of the various ways in which geometric reasoning could (or should) be leveraged
    - the "object of formalism" and its accompanying difficulties for students
- Identified 54 papers, with 36 of being of sufficient quality for further consideration

Studies of student reasoning → frameworks and methodological tools: Two examples

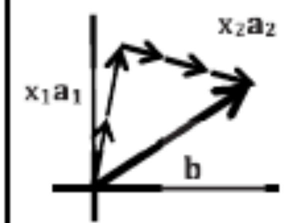
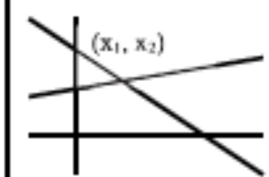
- A**x** = **b** (Larson & Zandieh, 2013)
- The invertible matrix theorem (Selinski, Rasmussen, Wawro, & Zandieh, 2014)

Studies of mathematicians: One example

- Eigenvectors (Sinclair & Tabaghi, 2010)

How do you symbolically and geometrically interpret or make sense of A**x** = **b**?

The framework's power is in its potential to help teachers, researchers, and curriculum designers better understand ways of supporting students in developing the ability to move flexibly among interpretations to powerfully leverage the analytic tools of linear algebra.

Interpretation of A**x**=**b**	Symbolic Representation	Geometric Representation
Linear combination (LC) interpretation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$	A: set of column vectors $(\mathbf{a}_1, \mathbf{a}_2)$ **x**: weights (x_1, x_2) on column vectors of A **b**: resultant vector	
System of equations interpretation $a_{11}x_1 + a_{12}x_2 = b_1$ $a_{21}x_1 + a_{22}x_2 = b_2$	A: entries viewed as coefficients $(a_{11}, a_{12}, a_{21}, a_{22})$ **x**: solution (x_1, x_2) **b**: two real numbers (b_1, b_2)	
Transformation interpretation $T:\mathbf{x}\mapsto\mathbf{b}$, $T(\mathbf{x})=A\mathbf{x}$	A: matrix that transforms **x**: input vector **b**: output vector	

Larson & Zandieh (2013)

Making connections – the invertible matrix theorem

(Selinski et al., 2014)

Suppose you have a 3×3 matrix A, and you know that A is invertible. Decide if each of the following statements is true or false, and explain your answer.

(i) The column vectors of A are linearly independent.

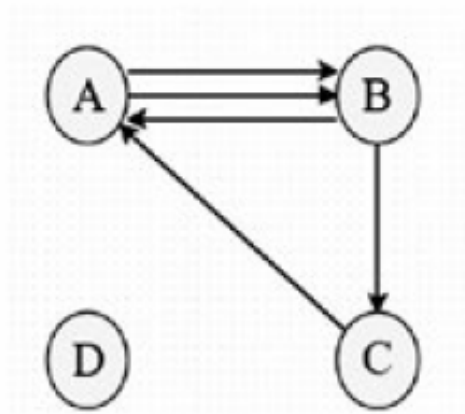
(ii) The determinant of A is equal to zero.

(iii) The column vectors of A span $\mathbf{R}^3$.

(iv) The null space of A contains only the zero vector.

(v) The row-reduced echelon form of A has three pivots.

Sample Digraph

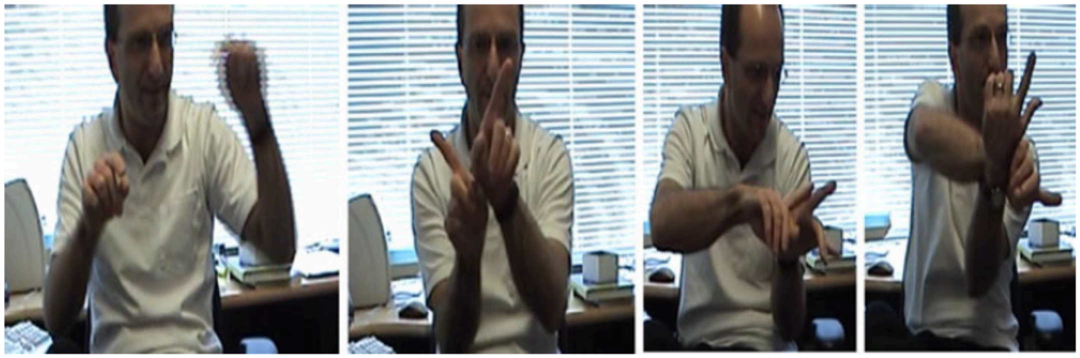


Adjacency Matrix M

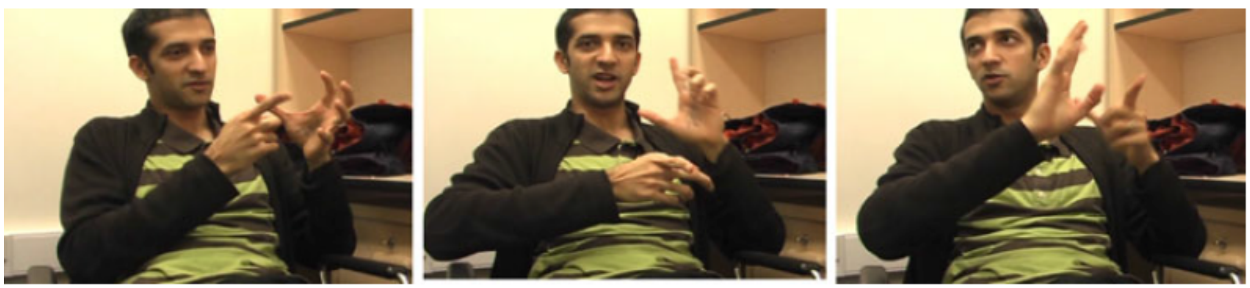
	A	B	C	D
A	0	2	0	0
B	1	0	1	0
C	1	0	0	0
D	0	0	0	0

- The method makes use of mathematical constructs from digraph theory, such as walks and being strongly connected, to indicate possible chains of connections and flexibility in making connections within and between concepts.
- The authors Illustrate the usefulness of this method for comparing differences in the structure of the connections, as exhibited in what they refer to as **dense, sparse, and hub** adjacency matrices.
- Another contribution of the adjacency matrix method is that it requires the construction of a conceptually structured inventory of students' conceptions.

How mathematicians understand eigenvectors and eigenvalues – An embodied cognition analysis (Sinclair & Tabaghi, 2010)



a. move output vector b. move vector around c. vectors line up d. chasing vectors



a. right hand showing unit circle b. vectors bending and twisting c. vector not changing direction

- Found a prevalence of metaphorical language and gesturing to convey vectors as objects in space that get mapped to their scalar multiples
- Gesture offers more possibility than spoken language for expressing continuity, time and motion

Part 2 - Research on Teaching

- The 2007 Handbook chapter contained little to no review of undergraduate mathematics teaching, which was a reflection of the state of the field.
- Today situation is quite different – we identified nearly 40 empirical studies that focused on instruction.
  - Research that examines lecture-oriented instruction
  - Research that examines inquiry-oriented instruction

Highlight 3 studies on lecture-oriented instruction

- Artemeva and Fox
- Virman
- Lew et al

Highlight 3 studies of inquiry-oriented instruction

- Small scale study in DES
- Large scale study - Freeman et al
- Laursen et al

Switch from Post-Calculus to a US national study of Calculus

A Cultural Shift

Lynn Steen (2011, p. 5) in his contribution to the *Project Kaleidoscope 20th Anniversary Essay Collection* writes the following:

> Professional meetings of university mathematicians, which in the mid-1980s were predominantly devoted to mathematical research and applications, are today a nearly equal mix of mathematics and mathematics education. For a community steeped in a tradition that focused only on research and exposition of mathematics, the very visible emphasis on teaching and learning is a major change in the culture.

Lecture-oriented instruction

Artemeva and Fox (2011) provide a comprehensive portrait of the writing and talking that occurs in lectures.

- Informed by rhetorical genre studies and communities of practice
- Analyzed 50 different lecture classes from different cultures and content
- Identified the genre they call "Chalk Talk"

- Chalk talk practices include
  - verbalizing everything written on the board,
  - metacommentary about what was written,
  - board choreography,
  - using pointing gestures to highlight key issues, relationships
  - using rhetorical questions to signal transitions, reflection, or to check for understanding.

Lecture-oriented instruction

Using Sfard's (2008) commognitive approach, Viirman (2014) analyzed the lectures of seven different Swedish university mathematics instructors.

- The overall findings support Arteva and Fox's (2011) delineation of the practices that comprise "chalk talk" but also explore in more depth differences between the seven lecturers in the way in which doing mathematics is modeled for learners.
- For example, Viirman detailed differences in the lecturers' routines for constructing definitions
  - By stipulation, which introduces a new concept via a definition.
  - By "saming." In this routine, several examples are presented and then the definition comes out of an examination of what property unites them.

Lectures in advanced mathematics
(Lew, Fukawa-Connelly, Meija-Ramos, & Weber, 2016)

- Case study– One professor (Dr. A) with 30 years experience and an excellent reputation as a real analysis instructor
- One 11-minute proof that a sequence $\{x_n\}$ with the property that $|x_n - x_{n+1}| < r^n$ for some $0<r<1$ is convergent
- Interviews with three pairs of students

- Instructor shown video of his lecture and interviewed about his goals
  - First asked to describe why he presented this proof to students
  - Then asked to stop the video recording at every point he thought he was trying to convey mathematical content

Student learning from lecture

Three student pairs were interviewed with four passes

Pass 1: Students recalled what they learned from the proof by reviewing their notes.

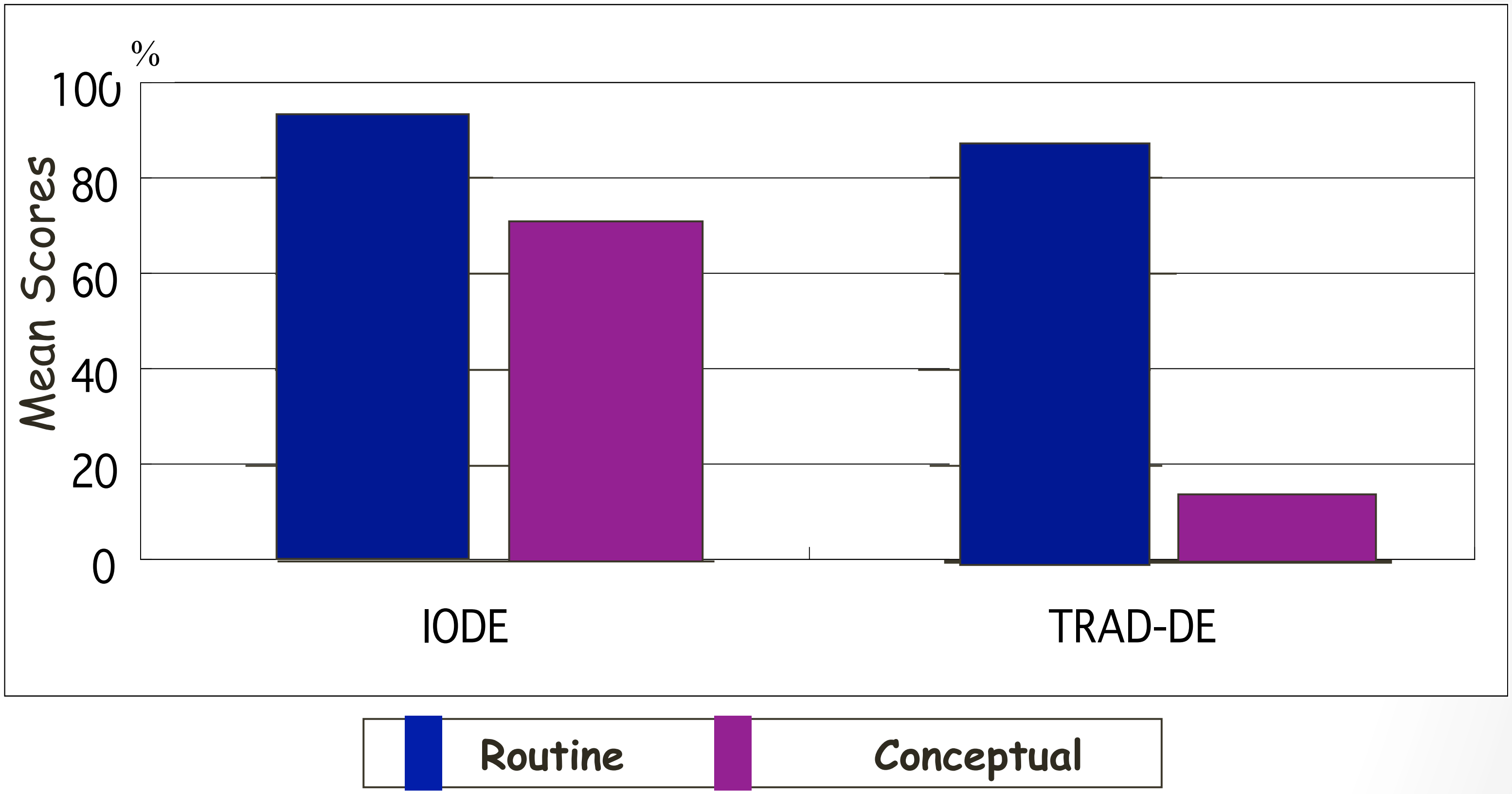
Pass 2: Students watched the lecture again in its entirety, took notes, and were asked what they learned and what they thought the instructor was trying to convey.

Pass 3: Students were shown short specific clips of the video and asked what they thought the professor was trying to convey.

Pass 4: Students were asked whether particular content highlighted by Dr. A in his interview could be gleaned from the proof they just watched.

Content conveyed by professor	Pair #1	Pair #2	Pair #3
To show sequence is convergent without a limit candidate, show it is Cauchy	Pass 3	Pass 3	Never
Triangle inequality is important for proofs in real analysis	Pass 2	Pass 3	Pass 3
Geometric series in one’s “toolbox” for working with bounds and keeping quantities small	Never	Never	Never
How to set up proofs to show a sequence is Cauchy	Pass 4	Pass 2	Pass 4
Cauchy sequences can be thought of as “bunching up”	Pass 3	Pass 3	Pass 3

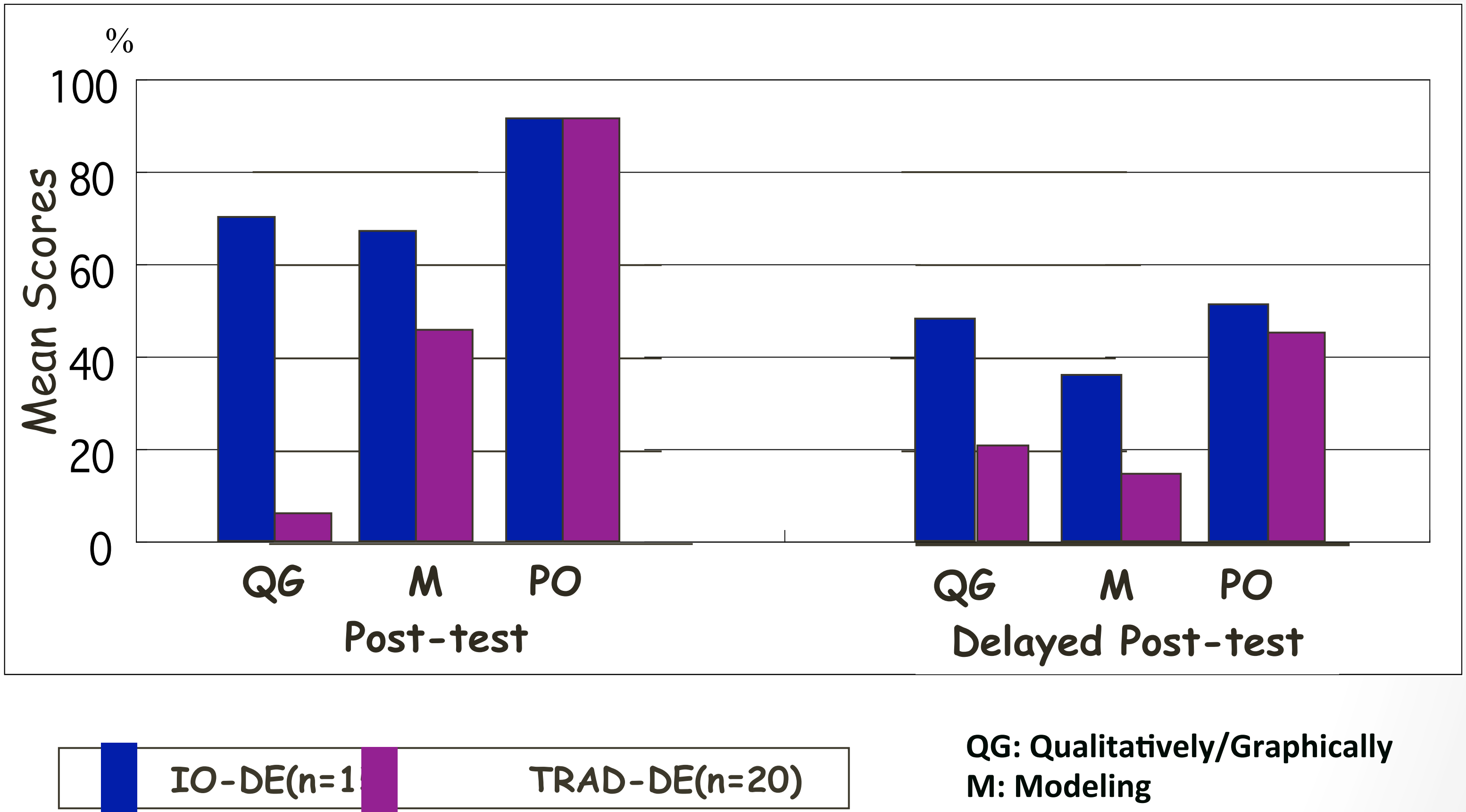
Inquiry-Oriented Instruction – Small Scale Study in DEs



- 4 different sites, N = 111

Rasmussen & Kwon (2007)

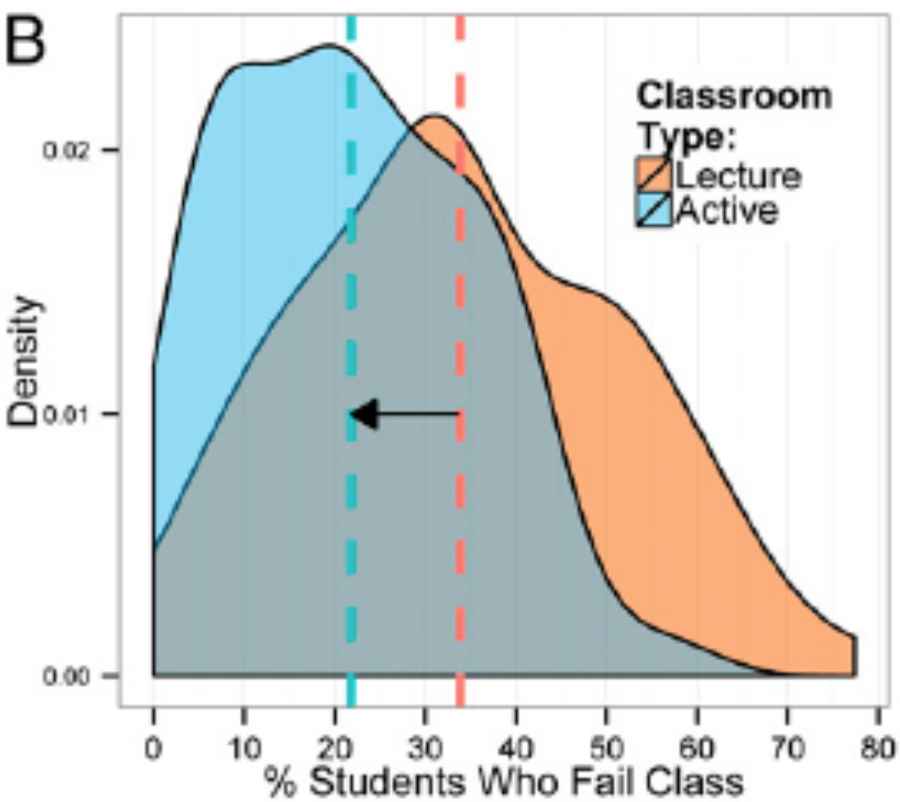
Students' retention of mathematical knowledge and skills in differential equations



QG: Qualitatively/Graphically
M: Modeling
PO: Procedurally Oriented

Inquiry-Oriented Instruction – Large Scale Study

Freeman et al. (2014) examined 225 studies that compared student achievement in a range of undergraduate STEM courses and found that students in lecture-oriented classes were 1.5 times more likely to fail than were students in inquiry-oriented classes.



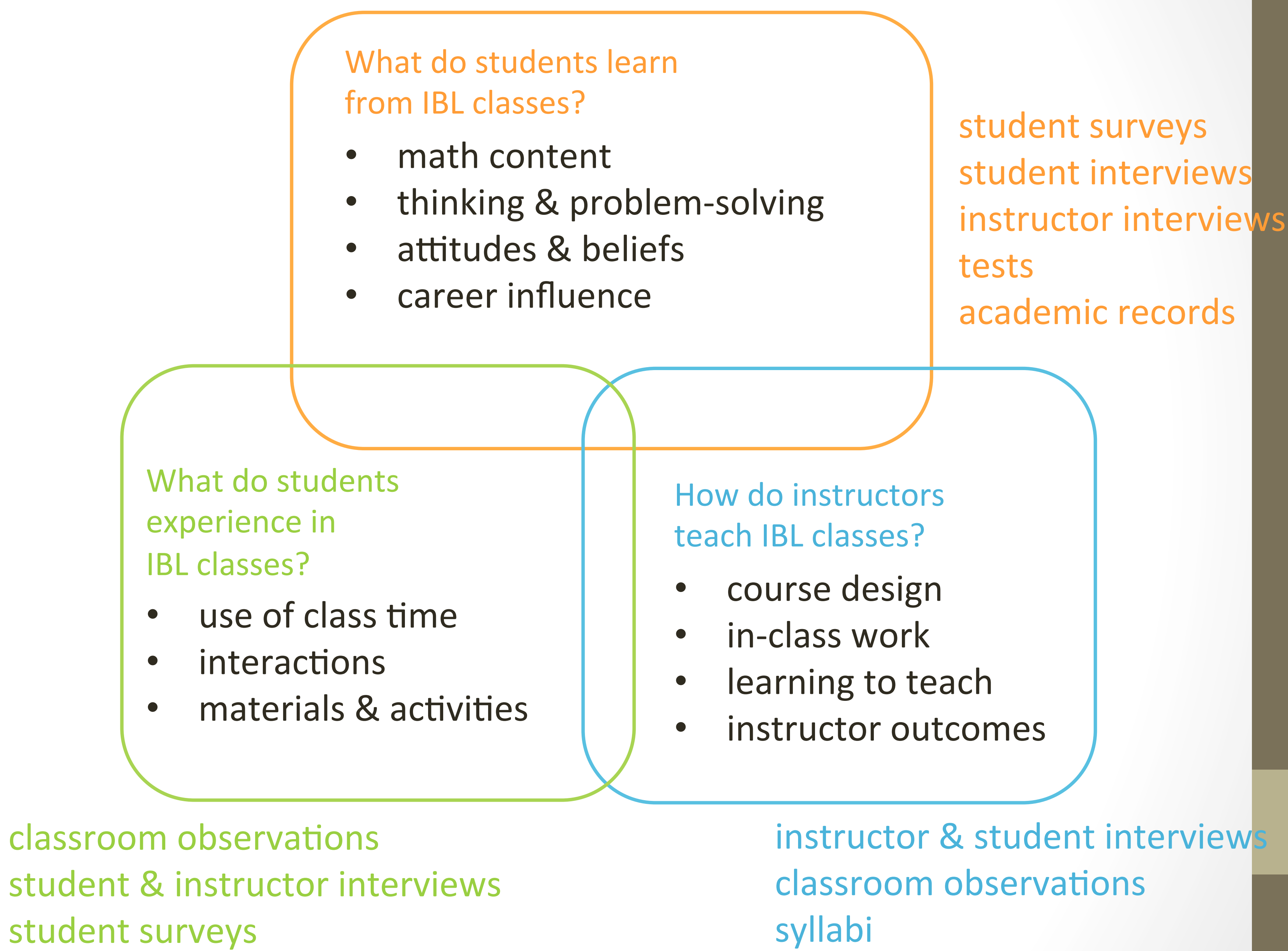
A provocative conclusion

"If the experiments analyzed here had been conducted as randomized controlled trials of medical interventions, they may
have been stopped for benefit—meaning that enrolling
patients in the control condition might be discontinued because the treatment being

Laursen et al (2014)

Study sites: 4 Inquiry Based Learning (IBL) Math Centers at Top Research Universities and ~30 courses

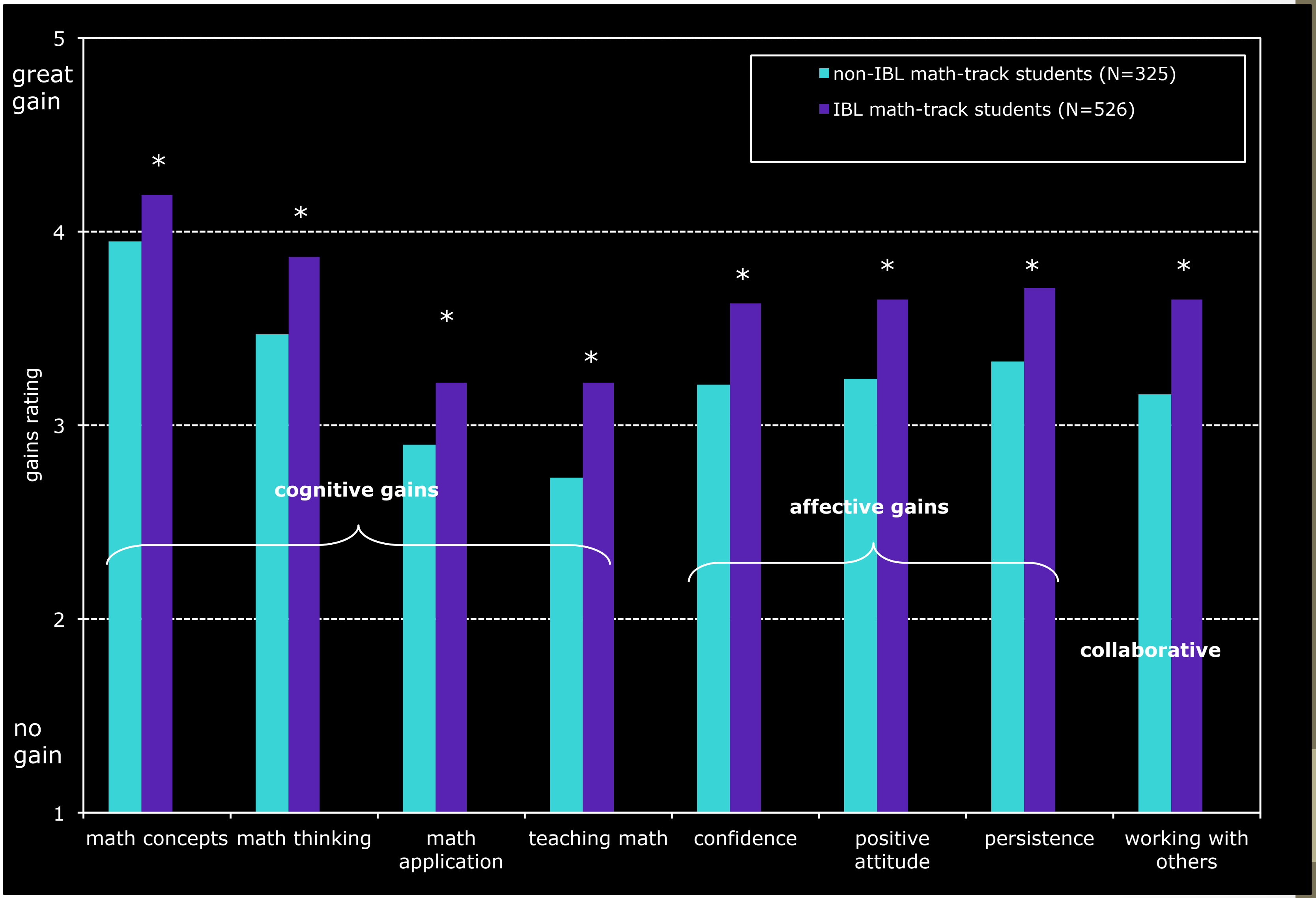
comparison non-IBL math-track courses	IBL math-track courses	
comparison non-IBL math-track courses	IBL math-track courses	
comparison non-IBL math-track courses	IBL math-track courses	IBL pre-service teacher courses
	IBL math-track	IBL pre-service


What do students learn from IBL classes?
• math content
• thinking & problem-solving
• attitudes & beliefs
• career influence
student surveys
student interviews
instructor interviews
tests
academic records
What do students experience in IBL classes?
• use of class time
• interactions
• materials & activities
classroom observations
student & instructor interviews
student surveys
How do instructors teach IBL classes?
• course design
• in-class work
• learning to teach
• instructor outcomes
instructor & student interviews
classroom observations
syllabi

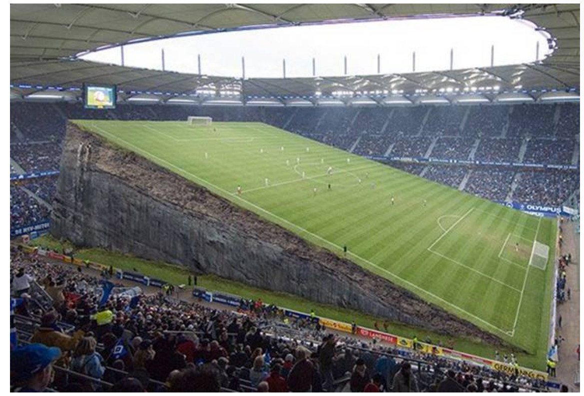
What do students learn from IBL classes?

1. IBL instruction has positive outcomes fo students
2. Especially women
3. And students with lower prior math achievement

Learning gains: from survey, post-only



Gender differences?



Laursen et al. (2014) report the following:

- In non-Inquiry courses, women reported gaining less mastery than did men, but these differences vanished in IBL courses.
- “That this apparent deficit can be so readily erased shows that its cause is not a deficit among female students, but rather that non-inquiry courses do selective disservice to women. That is, inquiry-oriented methods do not “fix” women but fix an inequitable course.”

Summary of findings, IBL vs. non-IBL courses

IBL students report higher learning gains on surveys…

- cognitive (math thinking, understanding concepts, applying math knowledge, teaching)
- affective (confidence, positive attitude, persistence)
- collaborative gains (working with other students)
- Interviews corroborate the nature of gains reported on surveys

IBL students get grades as good or better than those of non-IBL students **in later courses**

IBL students' attitudes & beliefs are modestly more supportive of learning following a course (compared to non-IBL students)

Summary of findings – low and high achievers

IBL low achievers earn better grades after an IBL course (even though grades decline for all others)

IBL low achievers report higher learning gains
- compared with high achievers & with non-IBL peers
- especially pre-service teachers

(no differences for IBL vs non-IBL high achievers)

High achievers who take an IBL course early in their UG career take more math courses than non-IBL peers

(low achievers do not)

No harm to high achievers (& they may take more courses)

Characteristics of Successful Programs in College Calculus: USA National Study

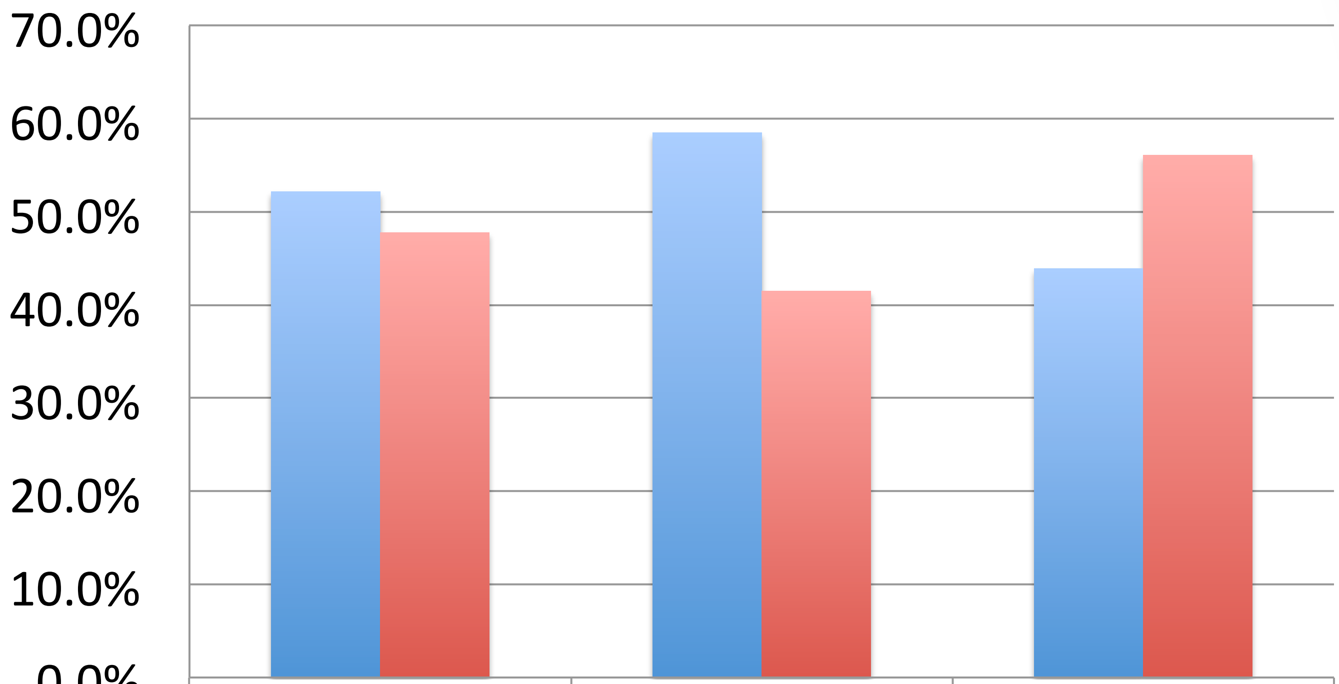
Phase I: Six web-based surveys to identify factors that are correlated with success in Calculus I

207 two-year colleges	→	40 (19%) participated
134 undergraduate colleges	→	41 (31%) participated
60 master's universities	→	21 (35%) participated
120 research universities	→	66 (55%) participated

Phase II: Case studies of 16 <u>successful</u> calculus programs

Bressoud, Mesa, & Rasmussen (2015)

Phase 1 survey findings in Calculus I



	Overall	STEM intending	Switchers
Male	52.2%	58.5%	43.9%
Female	47.8%	41.5%	56.1%

Total	4690	3173	478

Instructor Pedagogy: Factor analysis “Good Teaching” and “Ambitious Teaching”

“Good Teaching”

My Calculus Instructor:

- listened carefully to my questions and comments
- allowed time for me to understand difficult ideas
- presented more than one method for solving problems
- asked questions to determine if I understood what was being discussed
- discussed applications of calculus
- encouraged students to seek help during office hours
- frequently prepared extra material
- Assignments were challenging but doable
- My exams were graded fairly
- My calculus exams were a good assessment of what I learned

Instructor Pedagogy: Factor analysis "Good Teaching" and "Ambitious Teaching"

"Ambitious Teaching"

My Calculus Instructor:

- Required me to explain my thinking on homework and exams
- Required students to work together
- Had students give presentations
- Held class discussions
- Put word problems in the homework and on the exams
- Put questions on the exams unlike those done in class
- Returned assignments with helpful feedback and comments

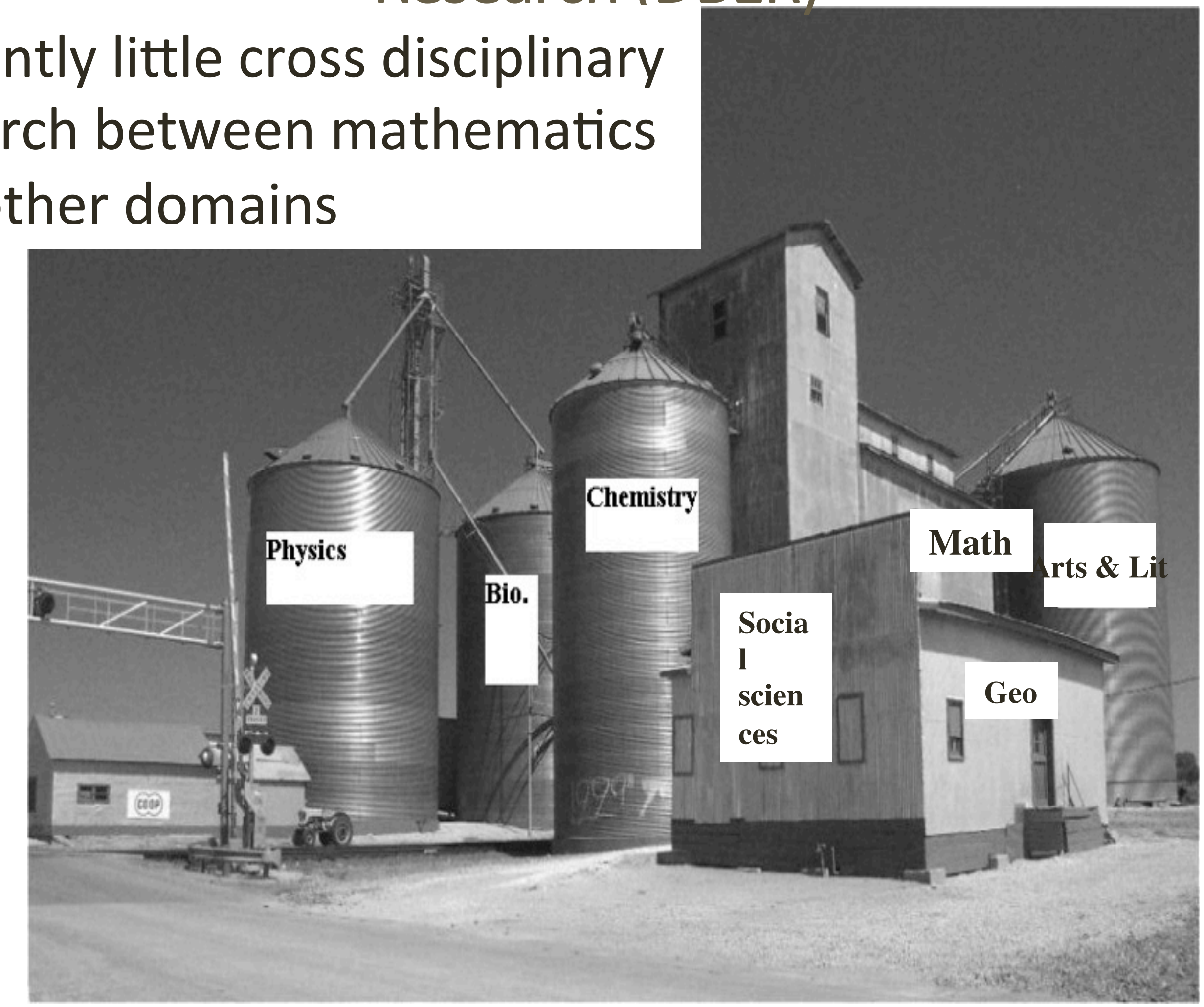
Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring Relationships Between Setting Up Complex Tasks and Opportunities to Learn in Concluding Whole-Class Discussions in Middle-Grades Mathematics Instruction. *Journal for Research in Mathematics Education*, *44*(4), 646–682.

Switcher Rates for Low and High Levels of Good and Ambitious Teaching

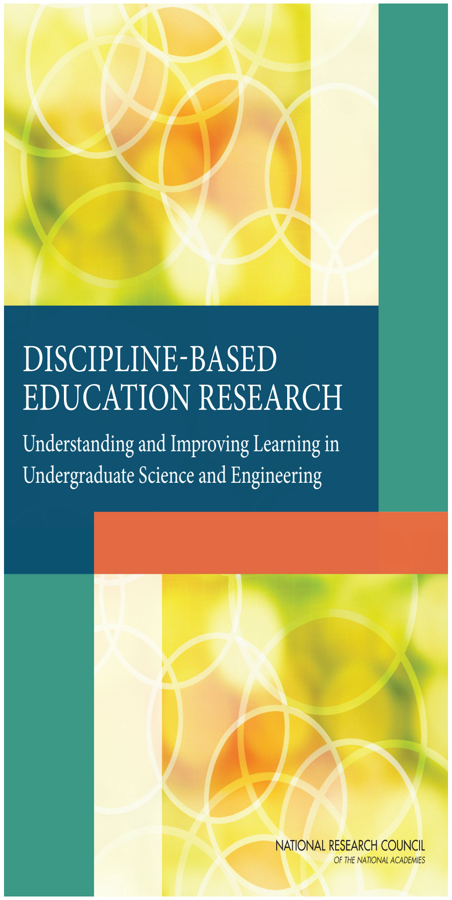
	Good Teaching Low	Good Teaching High
Ambitious Teaching Low	16.2%	10.4%
Ambitious Teaching High	11.9%	7.0%

Part 3 – Future direction: Connecting RUME and other Discipline Based Educational Research (DBER)

Currently little cross disciplinary research between mathematics and other domains



Slide courtesy of Susan Singer



National Research Council (2012)

- DBER Investigates teaching and learning using a range of methods with <u>deep grounding in the discipline's priorities, worldview, knowledge, and practices</u>
- Grounded in science and engineering disciplines
- Informed by and complementary to
  - Cognitive science
  - Educational psychology
  - K-12 education research

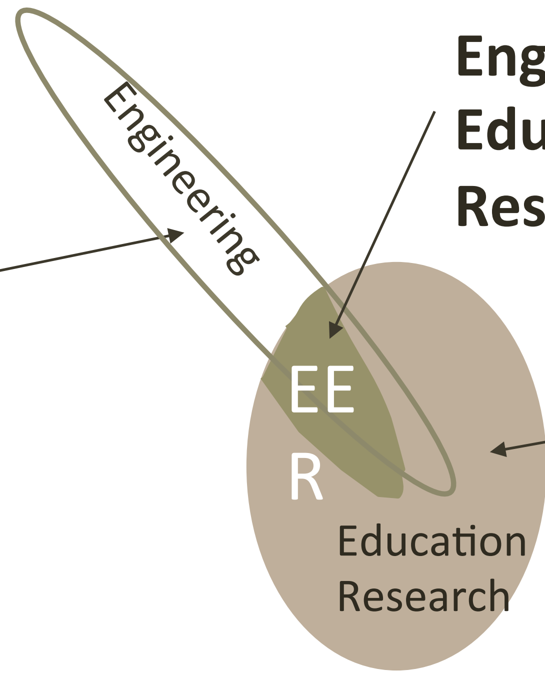
Key goals for DBER research

- Understand **how people learn** the concepts, practices, and ways of thinking in science, engineering, and mathematics.
- Understand the nature and development of **expertise** in a discipline and how this differs across disciplines.
- Help to **identify and measure appropriate learning objectives** and **instructional approaches** that advance students toward those objectives.
- Contribute to the knowledge base in a way that can guide the **translation of DBER findings** to classroom practice.
- Identify approaches to make science and engineering **education broad and inclusive.**

New STEM DBER Alliance to connect disciplines

Discipline-Based

- Content
- Culture
- Priorities

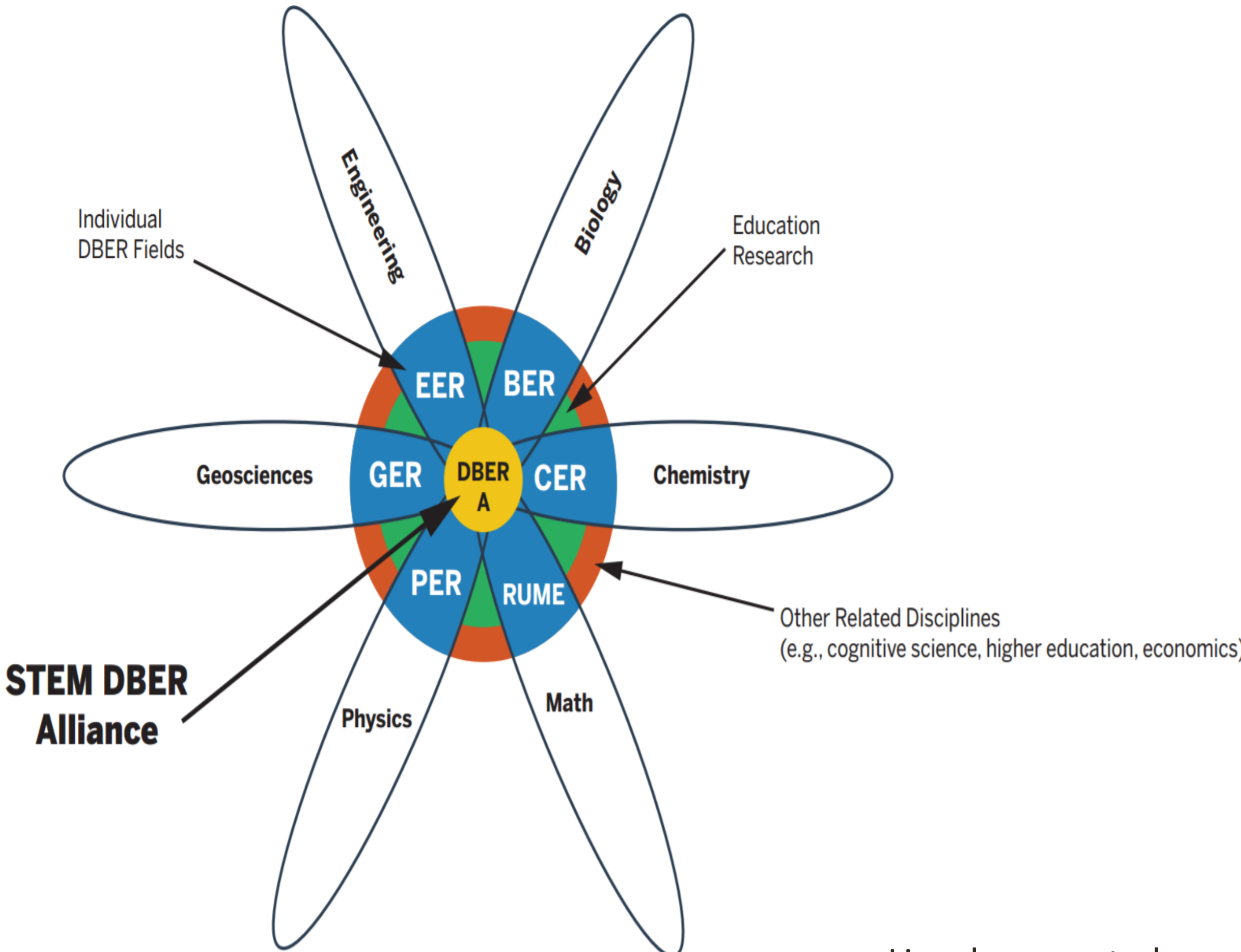


Engineering Education Research

Education Research

- Topics
- Methods

The STEM DBER Alliance (DBER-A)



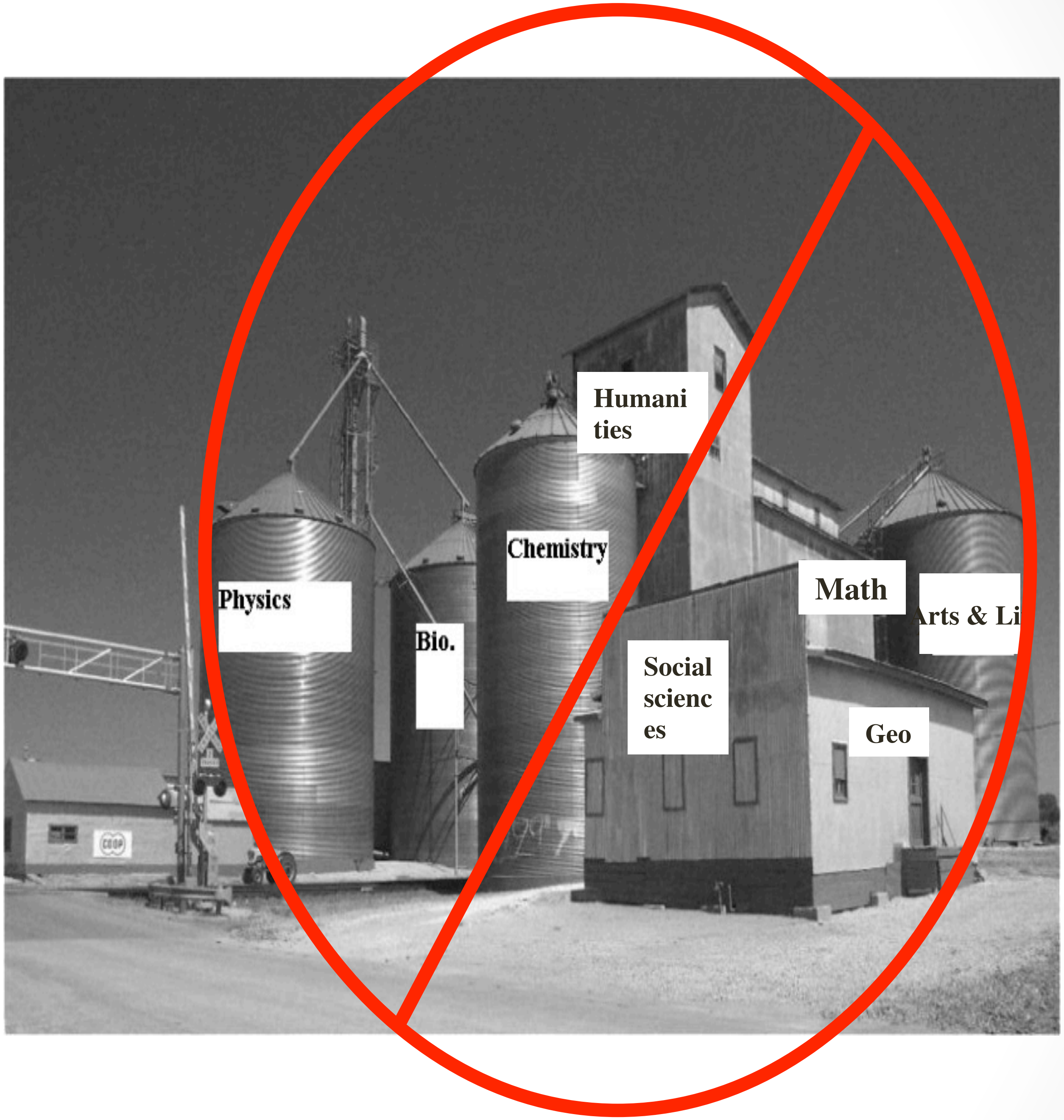
Henderson et al. (2017)

Connecting STEM research areas

Facilitated by DBER Alliance ⟷ Situated in DBER Alliance

A: Develop Understanding of other Contexts	B: Transfer of Research Ideas/Methods	C: Collaborative Research	D: Cross-Cutting Research	E: Research Community Development
• Discipline 2 requires understanding of Discipline 3 to improve work in Discipline 2. • **Example:** How to develop a physics course for biology majors	• Discipline 1 learns ideas and approaches from Discipline 2 to improve work within Discipline 1. • **Example:** How to study problem solving	• Disciplines 3 and 4 collaborate on cross-disciplinary research that improves work in both Disciplines. • **Example:** How the teaching of "energy" be coordinated across multiple STEM disciplines	• Disciplines 4, 5, and 6 collaborate on research that spans and improves all STEM disciplines. Disciplines 1, 2, and 3 also benefit from this. • **Example:** Improving inclusion and diversity	• Multiple disciplines interact to set norms (implicit or explicit) for DBER research. DBER (and all Disciplines) benefit. • **Example:** How student learning gets reported

Math modeling holds much promise for breaking down silos



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Chris Rasmussen

chris.rasmussen@ sdsu.edu